OR Lecture 4

27 January 2021 23:23

Linear Programming

$$\boxed{\begin{aligned}&\min/\max\ f(x)\\ &\text{s.t. } g_i(x)\le 0,\ i\in\{1\ldots n\}\\ &\quad x\in S\subseteq \mathbb{R}^n\end{aligned}}$$

$g_i:\mathbb{R}^n\to\mathbb{R}$ — AFFINE

$f:\mathbb{R}^n\to\mathbb{R}$ → LINEAR

Why does f being linear suffice? An affine $f(x)$ can be expressed as $f(x)=g(x)+c$ where $g(x)$ is linear. It is easy that the value of x that max/minimizes $f(x)$ will be the one that max/minimizes $g(x)$. Thus the two optimization problems are equivalent.

A general linear program looks like this:

$$\min/\max\ c^Tx$$
$$\text{s.t. } a^{i^T}x-b_i\le 0\quad \forall i\in\{1\ldots m\}$$
$$x\in\mathbb{R}^n$$

$a^i\in\mathbb{R}^n$, $b_i\in\mathbb{R}$

$Ax-b\le 0 \Leftrightarrow Ax\le b$

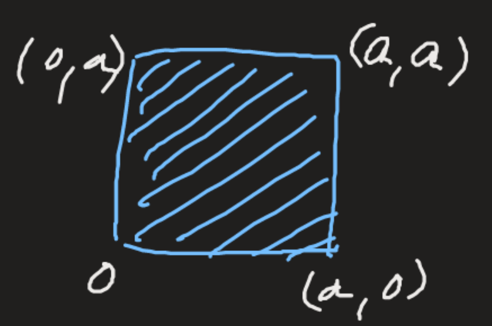
a^i are columns of A

b_i are elements of b

General form:
$$\boxed{\begin{aligned}&\min/\max\ c^Tx\\ &\text{s.t. } Ax\le b\\ &\quad x\in\mathbb{R}^n\end{aligned}}\qquad A\in\mathbb{R}^{m\times n},\ b\in\mathbb{R}^m$$

Feasible region of a linear program

$\{x\in\mathbb{R}^n : Ax\le b\}$ ← POLYHEDRON



In $\mathbb{R}^2$, $(-1)x+(0)y\le 0$

$(1)x+(0)y\le a$

$(0)x+(-1)y\le 0$

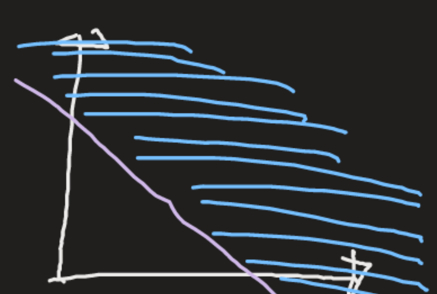
$(0)x+(1)y\le a$

$$\Rightarrow \underbrace{\begin{bmatrix}-1&0\\1&0\\0&-1\\0&1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\y\end{bmatrix}}_{x}\le\underbrace{\begin{bmatrix}0\\a\\0\\a\end{bmatrix}}_{b}$$

Can a circle be described in the form $\{x\in\mathbb{R}^2: Ax\le b\}$ for some A and b? No, as its boundary cannot be described by using finitely many line segments/lines.

Defn: Intersection of a finite number of half-spaces is called a polyhedron.

Can a polyhedron be unbounded? Yes!



Defn: A bounded polyhedron is called a polytope.

Show that every polyhedron is a convex set.

Proof: Consider two points $x,y\in$ polyhedron $P=\{x\mid Ax\le b\}$

$\therefore Ax\le b,\ Ay\le b$

$A(\lambda x+(1-\lambda)y)=\lambda Ax+(1-\lambda)Ay\le\lambda b+(1-\lambda)b=b$

$\therefore \lambda x+(1-\lambda)y\in P\ \forall\lambda\in[0,1]$.

Thus every polyhedron is a convex set.

Note: Every polytope is a polyhedron. Not vice versa!

Consider the following linear program.

$$\min\ c^Tx\qquad A\in\mathbb{R}^{m\times n}$$
$$\text{s.t. } Ax=b\qquad b\in\mathbb{R}^n$$
$$x\in\mathbb{R}^n$$

Linear program $\not\equiv$ Linear system of eqns

$F=\{x\in\mathbb{R}^n: Ax=b\}$

Cases: (a) $F=\phi$ no optimal solution (LP is infeasible)

(b) F is a singleton i.e. $F=\{\bar{x}\}$ LP has unique optimal solution $\bar{x}$

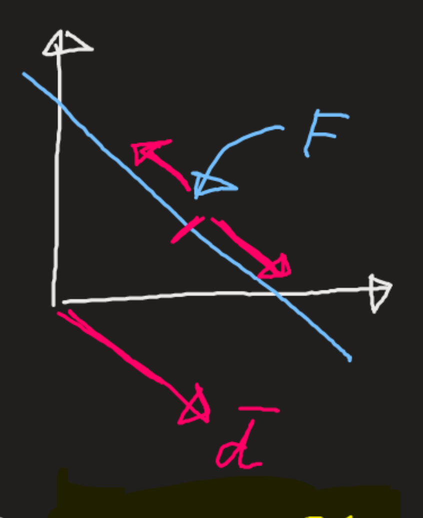
(A is $n\times n$ non-singular)

(c) F has infinitely many solutions. When does this happen?

The system of linear equations has infinitely many solutions if $\exists$ a non-trivial null space of A i.e. $\mathcal{N}(A)$.

x is a solution then $\forall d\in\mathcal{N}(A)$, $\alpha\in\mathbb{R}$, $(x: Ax=b)$

$\bar{x}=x+\alpha d\Rightarrow A(\bar{x})=Ax+\cancel{\alpha Ad}=b$



one solution ⟶ infinitely many solutions derived from that one solution

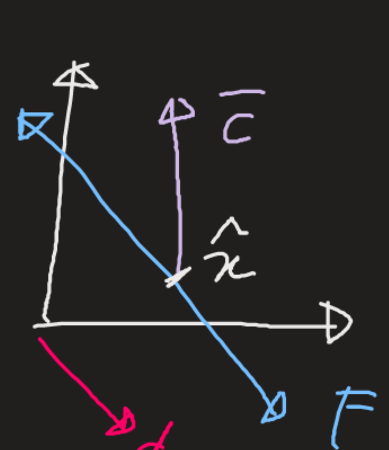
If $c\perp\mathcal{N}(A)$, entire F is optimal i.e. $\forall x\in F$, we obtain the optimal value $c^Tx=\beta^*$.

Consider $\hat{x}\in F$. $\therefore F=\{x:\hat{x}+\alpha\bar{d},\alpha\in\mathbb{R}\}$.

Now for any $x\in F$, $c^Tx=c^T(\hat{x}+\alpha\bar{d})=c^T\hat{x}+\cancel{c^T\alpha\bar{d}}$ why?

$\therefore c^T\hat{x}$ is the optimal value β^* and every point on F is an optimal solution.

What if $c\not\perp\mathcal{N}(A)$?



Any point $x\in F$, $x=\hat{x}+\alpha\bar{d}$, $\alpha\in\mathbb{R}$.

$c^Tx=c^T\hat{x}+\alpha c^T\bar{d}$

Say our objective sense is that of maximisation.

If $c^T\bar{d}<0$, make $\alpha\to-\infty$ and $c^Tx\to\infty$

If $c^T\bar{d}>0$, make $\alpha\to\infty$ and $c^Tx\to\infty$

In either case, the problem is UNBOUNDED.