OR Lecture 1

13 January 2021 12:39

Introduction to Optimization

Defining a mathematical optimization problem:

$$\max/\min \; f(x) \leftarrow \text{objective function}$$
$$\text{s.t. } g_i(x) \le \alpha_i \;\; \forall i \in I \leftarrow \text{constraints}$$
$$x \in \mathbb{R}^n \text{ or } x \in S \subseteq \mathbb{R}^n$$

objective sense

decision variables

$g_i, \alpha_i \leftarrow$ parameters of the problem

Def^n: $F := \{x \in \mathbb{R}^n : g_i(x) \le \alpha_i, \forall i \in I\}$ is called the feasible set/region of the mathematical programming problem (MP)

$\bar{x} \in F \rightarrow$ feasible solution, $\bar{x} \notin F \rightarrow$ infeasible solution

Def^n: If $F = \phi$, then we say that the MP problem is infeasible
$F \ne \phi$, then the MP problem is feasible.

Def^n: If $\exists x \in F$ s.t. $f(x) \ge \lambda \;\forall \lambda \in \mathbb{R}$, then the MP problem is unbounded. (WLG assume maximization)

eg $\max\limits_{x \ge 0} x$ over $x \in \mathbb{R}$ is unbounded.

Def^n: (Bounded set) If $\exists r > 0$, s.t. $F \subseteq B(0, r)$ then F is said to be bounded. ← open ball

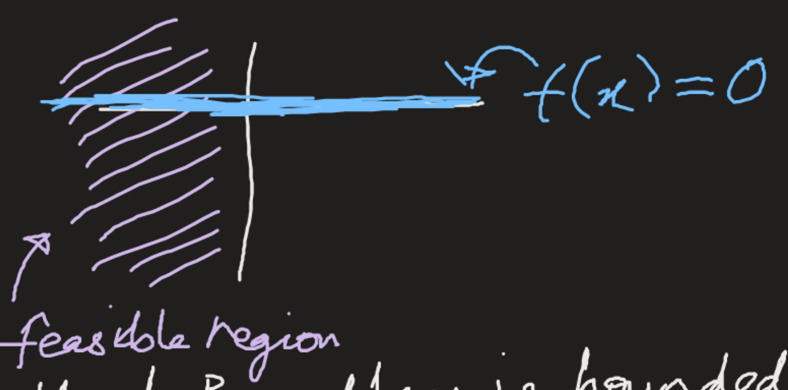
Remark: A LP problem cannot be unbounded unless its feasible region is unbounded.

F bounded $\Rightarrow$ LP problem bounded

What about the converse?

Here's an example that the converse needn't be true.

Consider the problem $\max\limits_{x \le 0} f(x)$ over $x \in \mathbb{R}$



$f(x)$ is bounded on $\mathbb{R} \longrightarrow$ hence the LP problem is bounded.

However, the feasible region for the constraint $x \le 0$ is the illustrated half plane, which is clearly unbounded.

Hence the converse is not true!

Optimal solutions and their existence

Def^n (Optimal solution)

(max) $\bar{x} \in F$ is called an optimal solution if $f(\bar{x}) \ge f(x) \;\forall x \in F$
(min) $\quad (f(\bar{x}) \le f(x))$

Def^n: (Local optimal solution)

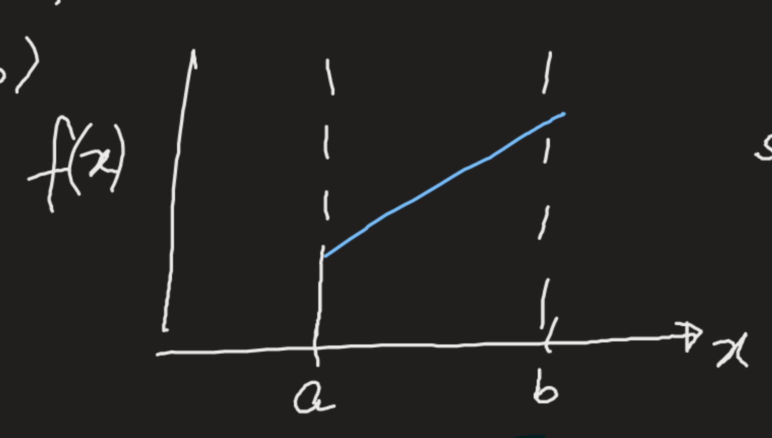
$\bar{x} \in F$ is called a local optimal solution if $\exists \varepsilon > 0$ such that $f(\bar{x}) \ge f(x) \;\forall x \in \boxed{N_\varepsilon(\bar{x}) \Rightarrow \|\bar{x} - x\| < \varepsilon}$ (max)
$(f(\bar{x}) \le f(x))$ (min)

Does every mathematical programming problem have an OPTIMAL solution?

1. $F = \phi$ (infeasible MP problem)
2. F is unbounded (necessary condition) (eg $\max\limits_{x \ge 0} x$ over $\mathbb{R}$)
3. Consider the following case

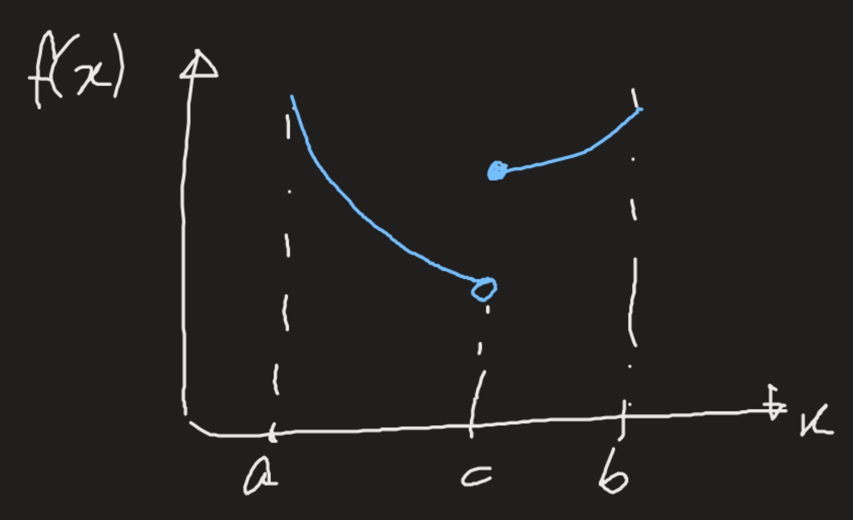
$F = (a, b)$

optimization over open sets



$\max\limits_{\text{s.t. } x \in F} x$ does not have an optimal solution

4. What if $f(x)$ is discontinuous over F?



$\min\limits_{x \in [a, b]} f(x)$ does not have an optimal solution

Def^n: Infimum of f over F, denoted as $\alpha = \inf\{f(x) : x \in F\}$ is the greatest lower bound on f over F i.e. $\alpha \le f(x) \;\forall x \in F$ and $\nexists \bar{\alpha} > \alpha$ and $\bar{\alpha} \le f(x) \;\forall x \in F$.

eg $\inf\{x : x > 0\} = 0$

Similarly, we define the supremum ($\sup\{f(x) : x \in F\}$) as the least upper bound on f over F.

Weierstrass Theorem (Fundamental theorem of optimization)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $F \subseteq \mathbb{R}^n$ be non-empty, bounded and closed. Then

$$\min\{f(x) : x \in F\} \text{ has an optimal solution.}$$

(sufficient)

Proof:

Consider $\alpha = \inf\{f(x) : x \in F\}$

Let $F^k = \{x \in F : \alpha \le f(x) \le \alpha + \varepsilon^k\}$, $0 < \varepsilon < 1$, $k = 1, 2, \ldots$

Consider a convergent subsequence $\{x_k\}_{k \in K}$, $x^k \in F^k$, $k = 1, 2, 3, \ldots$

$\lim\limits_{k \to \infty} \{x^k\} \longrightarrow \bar{x} \in F$ (as F is closed)

As f is continuous, $\lim\limits_{k \to \infty} f(x^k) = f(\bar{x}) = \alpha$

Thus $\bar{x}$ is optimal.