Note Title 2/9/2008

Basic Ordinary Diff. Equations.

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Motivation:

- Laws of physics when expressed in math. terms lead to ODEs.
- Certain Biological Systems also lead to systems of ODEs.
- Balance Equations governing Chemical Reactions (Law of mass action)

Example: (i) Equation of a simple harmonic oscillator:

$$y'' + y = 0 \qquad (1.1) \quad \text{(Linear)}$$

general Solution: $c_1 \cos t + c_2 \sin t$ (1.2)

(ii) Simple Pendulum: $y'' + \sin y = 0$ (1.3) (non linear)

or

$$\frac{d}{dt}\left(\frac{1}{2}y'^2 - \cos y\right) = 0$$

$$y'^2 - 2\cos y = E \tag{1.4}$$

Energy Equation obtained via one integration.

Further integration would involve Elliptic functions.

(iii) Equations arising in Ecological Systems.

- The Lotka - Volterra model for a two species ecology consisting of a predator and prey.

Predators = Sharks ; population denoted $x(t)$

Prey = Sardines ; population $y(t)$

Governing ODEs

$$\left.\begin{aligned} \frac{dx}{dt} &= -ax + bxy \\ \frac{dy}{dt} &= cy - dxy \end{aligned}\right\} \tag{1.5}$$

a, b, c, d are positive constants.

From (1-5) we get

$$\frac{dx}{dy} = \frac{x(-a+by)}{y(c-dx)} \qquad (1.6)$$

Equation (1-6) is an example of a Variable Separable Equation.

Separating Variables in (1.6) and Solving,

$z = \phi(x,y)$

$z = E$

$$\left(\frac{c}{x} - d\right)dx = \left(b - \frac{a}{y}\right)dy$$

$$\therefore \quad c\ln x - dx = by - a\ln y + E$$

$$\phi(x,y) = c\ln x + a\ln y - dx - by = E \qquad (1.7)$$

The Constant E appearing in (1.7) is

the constant of integration. A solution containing constants of integration is called the <u>general</u> <u>solution</u>. Given the values of x and y at a particular time say $t=0$, one can evaluate E and we get a <u>particular</u> <u>solution</u>.

<u>Note:</u> (1-7) is the general solution of the DE (1-6)

We have still not completely solved (1-5). To solve (1-5) would mean finding x and y

as functions of t. We have found x as an implicit function of y or vice-versa through (1-7).

We take up one more example of a variable separable equation.

Consider

$$x(y^2+1)\,dy - (y^2-1)(x^2+1)\,dx = 0$$

which may be rewritten as

$$\left(\frac{y^2+1}{y^2-1}\right)dx - \left(x+\frac{1}{x}\right)dx = 0$$

The integration of this equation is left as an easy exercise for the student.

Finding ODEs for a family of Curves.

Consider the family of Circles whose centers lie on the x-axis and touch the y-axis:

Equation of a typical circle in the family is

$$(x-c)^2 + y^2 = c^2$$

or

$$x^2 + y^2 - 2cx = 0 \qquad (1.8)$$

Equation (1.8) contains one arbitrary constant c. Let us find the ODE whose general solution is (1.8).

Procedure: Diff. the equation (1.8) to obtain

$$2x + 2yy' - 2c = 0 \qquad (1.8)'$$

Now eliminate c between (1.8) and (1.8)' to get

$$x^2 + y^2 - x(2x + 2yy') = 0$$

$$\therefore (y^2 - x^2) - 2xy\frac{dy}{dx} = 0 \qquad (1.9)$$

If there are two constants in the family i.e a two parameter family one differentiates twice:

Ex:

$$y^2 = mx^2 + cx \qquad (1.10)$$

Diff once

$$2yy' = 2mx + c \qquad (1.10)'$$

Diff again: $2y'^2 + 2yy'' = 2m$ $\quad (1.10)''$

Eliminate m, c between (1.10), $(1.10)'$ and $(1.10)''$ to get a 2nd order ODE.

Orthogonal trajectories: Let us go back to the family of circles

$$x^2 + y^2 - 2cx = 0 \qquad (1.11)$$

The orthogonal trajectory for the family (1.11) is the family of circles

$$x^2 + (y-\alpha)^2 = \alpha^2 \qquad \text{——} \quad (1.12)$$

Each circle (1.12) meets all the circles in (1.11) orthogonally and vice-versa.

Problem:

Given the family (1.11) how does one determine (1.12)?

Step I: Find the ODE for the family (1.11)

We have found this to be

$$(y^2 - x^2) - 2xy\frac{dy}{dx} = 0 \qquad (1.13)$$

Step II: The orthogal trajectories satisfy the ODE

$$(y^2 - x^2) + 2xy\frac{dx}{dy} = 0 \qquad (1.14)$$

Obtained from (1.13) by replacing

$\frac{dy}{dx}$ by $\frac{-1}{\left(\frac{dy}{dx}\right)}$

This must be so since two curves one picked from each family meet orthogonally so the product of their slopes at the point of intersection must be -1.

<u>Step III</u>: Solve the ODE (1.14) for the orth. trajectories. We ought to recover (1.12)

We now show how an equation of the type (1.14) which is <u>not</u> variable separable is solved.

Homogeneous Equations:

Recall that a function $f(x,y)$ is said to be homogeneous of degree λ if

$$f(tx, ty) = t^{\lambda} f(x,y)$$

<u>Exercise</u>: Check that if f is homogeneous of degree λ then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = \lambda f(x,y) \qquad (1\cdot 15)$$

(Euler's Relation)

A differential Equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \qquad (1\cdot 16)$$

is said to be homogeneous if $M(x,y)$ and $N(x,y)$ are homogeneous of the <u>same</u> degree.

Example: The ODE (1.14)

$$(y^2 - x^2)\,dy + 2xy\,dx = 0$$

is homogeneous

(2) $$\left(\tan^{-1}\left(\frac{y}{x}\right)\right)(y^2 - x^2)\,dx - 2xy \log\left(\frac{y}{x}\right)dy = 0$$

is also a homogeneous ODE.

(3) $$(y^3 - 3x^2 y)\,dx + 4x^2y^2\,dy = 0$$

is <u>NOT</u> homogeneous.

Procedure for solving homogeneous ODEs:

The substitution $y = tx$ transform a homogeneous ODE to a Variable Seperable ODE.

We check this:

$$\frac{dy}{dx} = t + x\frac{dt}{dx}$$

Subst into

$$M(x,y)\,dx + N(x,y)\,dy = 0$$ we get

$$M(x, tx) + N(x, tx)\left(t + x\frac{dt}{dx}\right) = 0$$

$$\therefore \quad x^{\alpha} M(1,t) + x^{\alpha} N(1,t)\left(t + x\frac{dt}{dx}\right) = 0$$

$$\therefore \left(M(1,t) + tN(1,t)\right) + N(1,t)\,x\,\frac{dt}{dx} = 0$$

Which is clearly variable Seperable.

We now employ this method to solve

$$(y^2 - x^2)\,dy + 2xy\,dx = 0$$

put $y = tx$; $\frac{dy}{dx} = t + x\frac{dt}{dx}$

$$(t^2-1)\cancel{x^2}\left(t + x\frac{dt}{dx}\right) + 2\cancel{x^2}t = 0$$

$$(t^2-1)t + 2t + (t^2-1)\,x\,\frac{dt}{dx} = 0$$

$$\frac{dx}{x} + \left(\frac{t^2-1}{t^2+1}\right)\frac{dt}{t} = 0$$

$$\ln x + \ln t - \int \frac{2\,dt}{t(t^2+1)} = c$$

$$\ln(tx) - \int \frac{2t\,dt}{t^2(t^2+1)} = c$$

$$\therefore \ln y - \int \frac{d(t^2)}{t^2(t^2+1)} = c$$

$$\ln y - \ln\left(\frac{t^2}{1+t^2}\right) = c$$

$\therefore$ $\dfrac{y(1+t^2)}{t^2} = \tilde{c}$ (some other const)

$$\frac{y\left(1+ y^2/x^2\right)}{y^2/x^2} = \tilde{c}$$

$x^2+y^2 = \tilde{c}\,y$ which is of the form

$$\boxed{x^2 + (y-\alpha)^2 = \alpha^2}$$ Equation of the Orth. Trajectories

Exact Differential Equations / Integrating factors.

The diff. Eqn $M(x,y)\,dx + N(x,y)\,dy = 0$ (1.17)

is said to be exact if the planar vector field $M\hat{i} + N\hat{j} + 0\hat{k}$ is Conservative

That is if $\exists$ a scalar field $\phi(x,y)$ such that $M = \frac{\partial \phi}{\partial x}$ and $N = \frac{\partial \phi}{\partial y}$ (1.18)

Now suppose that (1.18) holds

Consider the function y given implicitly by the eqn

$$\phi(x,y) = C \qquad (1\text{-}19)$$

Diff (1.19)

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0$$

$$\therefore \quad M\,dx + N\,dy = 0$$

That is to say if (1.17) is exact and $\phi(x,y)$ is the scalar potential then the general solution of (1.17) is given implicitly by $\phi(x,y) = C$

<u>Theorem</u>: Suppose Ω is a <u>convex subset</u> of $\mathbb{R}^2$, $M(x,y)$ and $N(x,y)$ are smooth (continuously differentiable functions). The ODE

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad (1\text{-}20)$$

Check that $\quad y\,dx - x\,dy = 0 \qquad (1\text{-}21)$

is <u>NOT</u> exact. Condition (1-20) fails.

The proof of the theorem follows immediately from the fact that (1.20) is nothing but the condition

$$\text{Curl}\left(M\hat{i} + N\hat{j} + 0\hat{k}\right) = 0$$

and the domain is convex.

Now multiply (1.21) by $\dfrac{1}{x^2+y^2}$ and we see that the resulting equation

$$\left(\frac{y}{x^2+y^2}\right)dx - \left(\frac{x}{x^2+y^2}\right)dy = 0 \qquad (1.22)$$

is Exact. Check this!

Also find $\phi(x,y)$ such that

is

$$\phi_x = \frac{y}{x^2+y^2}, \qquad \phi_y = \frac{-x}{x^2+y^2}$$ on the right half plane.

<u>Example:</u> Q2 (i) from Tut Sheet p.3.

$$3x(xy-2)\,dx + (x^3+2y)\,dy = 0$$

$$M = 3x^2y - 6x$$
$$N = x^3 + 2y$$

$$\frac{\partial M}{\partial y} = 3x^2 = \frac{\partial N}{\partial x}.$$ The ODE is exact.

Let us find a $\phi(x,y)$ such that

$$\phi_x = M = 3x^2y - 6x \quad \text{——— (i)}$$
$$\phi_y = N = x^3 + 2y \quad \text{——— (ii)}$$

Integrating (i) w.r.t x

$$\phi = x^3y - 3x^2 + \psi(y) \quad \text{——— (iii)}$$

Where ψ is a function of y alone.

Diff. (iii) w.r.t y to get

$$\phi_y = x^3 + \psi'(y)$$

Comparing with (ii) we get

$$\psi'(y) + \cancel{x^3} = \cancel{x^3} + 2y$$

or $\psi'(y) = 2y$

$\psi(y) = y^2$ and

$$\phi(x, y) = x^3y - 3x^2 + y^2$$

The general solution of the ODE is

$$x^3y - 3x^2 + y^2 = C$$

Definition (Integrating Factors)

A function $\mu(x,y)$ is said to be an integrating factor for

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{if}$$

$$(M(x,y)\,\mu(x,y))\,dx + (N(x,y)\,\mu(x,y))\,dy = 0$$

is Exact

Thus $\dfrac{1}{x^2+y^2}$ is an integrating factor for $y\,dx - x\,dy = 0$ in the <u>first</u> <u>Quadrant</u>

<u>Note:</u> $\frac{1}{x^2}$, $\frac{1}{y^2}$ are <u>also</u> integrating factors for $y\,dx - x\,dy = 0$ in the first Quadrant.

Check that $\frac{1}{xy}$ is yet another!

Thus an ODE may have several essentially distinct integrating factors.

We now turn to:

Methods For finding Integrating Factors:

Let μ be an integrating factor for

$$M\,dx + N\,dy = 0$$

Then $$\frac{\partial}{\partial y}(M\mu) = \frac{\partial}{\partial x}(N\mu)$$

$$\therefore \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu = N\mu_x - M\mu_y$$

If there is an integrating factor which is a function of x - alone, i.e $\mu_y = 0$

then

$$\frac{\mu_x}{\mu} = \frac{1}{N}(M_y - N_x)$$

and $\exp\int \frac{1}{N}(M_y - N_x)$ is an i.f.

Likewise, if μ doesn't depend on x then

$$\frac{\mu_y}{\mu} = \frac{-1}{M}(M_y - N_x) \quad \text{and}$$

$\exp\int\left(-\frac{1}{M}(M_y - N_x)\right)$ is an i.f.

By Reversing the steps we derive:

<u>Theorem</u>: If $\frac{1}{N}(M_y - N_x)$ is a function of x alone,

$$\exp \int \frac{1}{N}(M_y - N_x)\, dx$$ is an i.f.

If $\frac{-1}{M}(M_y - N_x)$ is a function of y alone then

$$\exp \int \frac{-1}{M}(M_y - N_x)\, dy$$ is an i.f.

Let us now take up Q5(i) on page 4:

<u>Solve:</u> $y(8x - 9y)\,dx + 2x(x - 3y)\,dy = 0$

$$M = 8xy - 9y^2$$

$$N = 2x^2 - 6xy = 2x(x - 3y)$$

$$\frac{\partial M}{\partial y} = 8x - 18y; \quad \frac{\partial N}{\partial x} = 4x - 6y$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4x - 12y = 4(x - 3y)$$

$$\therefore \quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2}{x} \quad \left(\text{fu}^{n} \text{ of } x \text{ alone}\right)$$

Integrating Factor: $\exp \int \frac{1}{N}(M_y - N_x) = x^2$

Mult. ODE by x^2 to get

$$(8x^3y - 9y^2x^2)\,dx + (2x^4 - 6x^3y)\,dy = 0$$

Seek ϕ such that

$$\frac{\partial \phi}{\partial x} = 8x^3y - 9y^2x^2 \quad \text{——— (i)}$$

$$\frac{\partial \phi}{\partial y} = 2x^4 - 6x^3y \quad \text{——— (ii)}$$

$$\therefore \quad \phi = 2x^4y - 3y^2x^3 + \Psi(y)$$

$$\therefore \quad \phi_y = 2x^4 - 6x^3y + \Psi'(y)$$

Comparing w/ (ii)

$$\cancel{2x^4} - \cancel{6x^3y} + \psi'(y) = \cancel{2x^4} - \cancel{6x^3y}$$

$\therefore$ $\psi(y)$ is const.

Gen. Sol^n: $\quad 2x^4y - 3y^2x^3 = C.$

The linear First order ODE:

This is the equation

$$y' + p(x)y = Q(x) \qquad (1.23).$$

This can be rewritten as:

$$(Q(x) - y\,P(x))\,dx - dy = 0$$

$$\frac{\partial M}{\partial y} = -P(x) \quad ; \quad N = -1$$

$\therefore$ Integrating factor $\exp \int P(x)\,dx$

Mult the ODE through by this

$$\frac{d}{dx}\left(y \exp \int P(x)\,dx \right) = Q(x) \exp \int P(x)\,dx$$

$$\therefore \; y \exp \int P(x)\,dx = \int Q(x)\left(\exp \int P(x)\,dx\right) dx$$

Gen. Solution is

$$\boxed{y = e^{-\int p\,dx} \cdot \int Q(x)\, e^{\int p\,dx}\, dx \qquad (1\cdot 24)}$$

Note: There is no need to put const of integration after integrating $\int p\,dx$. because it would cancel out on RHS of (1·24)

Q8 (iii) on page 4 of Tut Sheet:

$$y' - 3y\tan x = 1.$$

$$P(x) = -3\tan x \quad ; \quad Q(x) = 1$$

$$\exp \int p\,dx = \cos^3 x$$

Solution $\quad y\cos^3 x = \int 1 . \cos^3 x\, dx$

$$\therefore \quad y = \frac{\sec^3 x}{4}\left(\frac{\sin 3x}{3} + 3\sin x + C\right)$$

The Bernoulli Equation

The Equation

$$y' + P(x)y = Q(x)y^n \qquad (1\text{-}25)$$

is called Bernoulli's ODE.

Cases: $n = 0, 1$ linear ODE

Assume $n \neq 0, 1$.

Divide (1-25) by y^n

$$y^{-n} y' + P(x)\, y^{1-n} = Q(x)$$

put $\quad y^{1-n} = u$

$$(1-n)\, y^{-n} y' = u'$$

$$\therefore \quad u' + (1-n) P(x)\, u = (1-n)\, Q(x) \qquad (1.26)$$

__Thm:__ The substitution $y^{1-n} = u$ reduces the Bernoulli's equation to a linear Eqn.

Example: Q9 (vii) on page 4 in Tut Sheet:

$$6y^2\,dx - x(2x^3 + y)\,dy = 0.$$

Equation is Bernoulli in x

$$\frac{dx}{dy} - \frac{x}{6y} = \frac{x^4}{3y^2}$$

Divide by x^4

$$x^{-4}\frac{dx}{dy} - \frac{x^{-3}}{6y} = \frac{1}{3y^2}$$

put $x^{-3} = u$, $x^{-4}\frac{dx}{dy} = -\frac{1}{3}\frac{du}{dy}$

$$-\frac{1}{3}\frac{du}{dy} - \frac{u}{6y} = \frac{1}{3y^2}$$

$$\frac{du}{dy} + \frac{u}{2y} = \frac{-1}{y^2} \quad ; \quad \text{i.f } \sqrt{y}$$

(Linear)

$$u\sqrt{y} = -\int \frac{1}{y^2}\cdot\sqrt{y}\,dy = c + \frac{2}{\sqrt{y}}$$

$$\therefore\ x^{-3} = \frac{2}{y} + \frac{c}{\sqrt{y}}$$

$$x^3 = y/(2 + c\sqrt{y})$$

End of Chapter – I

Picard's theorem will be taken up Later.

Miscellaneous Exercises:

(1) Find the orthogonal trajectories of the family

$$x^4 + y^4 - 6x^2y^2 = C$$

N.B: Try solving the ODE for orth. traj both as a homogeneous ODE as well as Exact ODE

(2) Consider $y' + 2xy = \cot\frac{1}{x}$; $y(1) = 1$.

Write out its solution as a definite integral and evaluate $\lim_{x\to\infty} y(x)$

(3) Solve $\frac{dy}{dx} - 3x^2 = x^2 e^{-y}$

(After mult by e^y put $e^y = u$). What about finding an i.f.

(4) Solve $y' = y(1-y)$; $y(0) = 0$ by finding the solution $y(t, \epsilon)$ of the problem

$$y' = y(1-y); \quad y(0) = \epsilon$$ and then

Computing $\lim_{\epsilon \to 0} y(t, \epsilon)$

(5) The ODE

$$3y\,dx - 2x\,dy + x^2 y^{-1}(10y\,dx - 6x\,dy) = 0$$

has an i-f of the form $x^h y^k$. Find it

(6) Find an integrating factor for

$$(x^3 + xy^4)\,dx + 2y^3\,dy = 0.$$

Chapter – II

Linear ODEs of 2nd or higher Order.

Generalities

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots + P_0(x)\, y = 0 \qquad (2.1)$$

is a linear ODE of nth order.

$P_j(x)$ are supposed to be continuous on an open interval I in the real line.

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots + P_0(x)\, y = Q(x) \qquad (2.2)$$

is a non-homogeneous linear n-th order ODE.

First, if $y_g(x)$ is the general solution of (2-1) and $y_p(x)$ is a particular solution of (2-2) then

$$y_g(x) + y_p(x)$$ is a solution of (2-2) and is the general solution.

This may be compared with the situation encountered in linear algebra where you solve

$$Ax = b \qquad (2\text{-}3)$$

Solve first $Ax = 0$ completely. That is find a basis $v_1, \dots, v_k$ for the null space of the matrix A

Find a particular solution x_0 of (2-3)

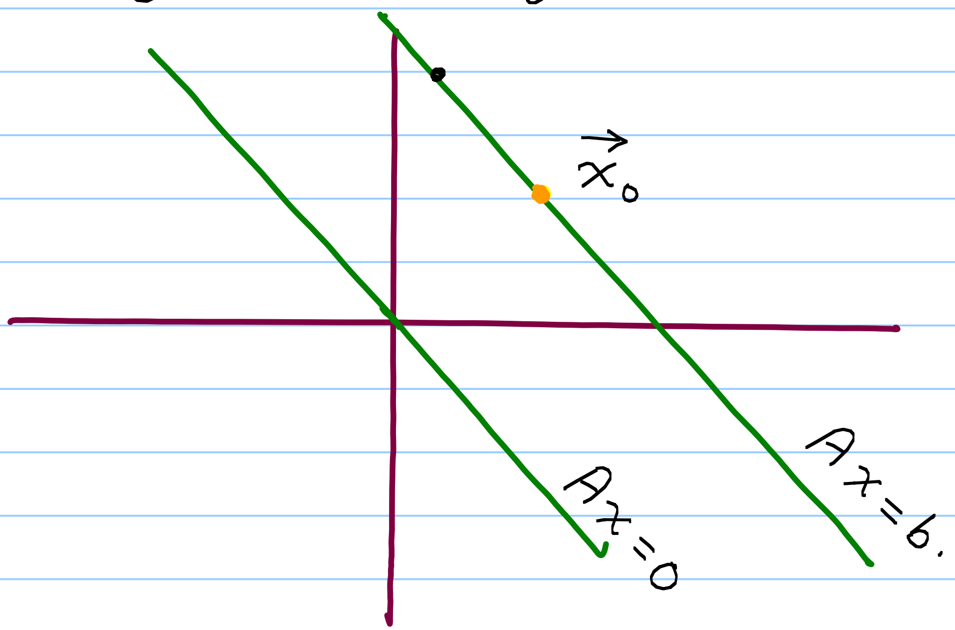
The complete solution of (2.3) is then

$$x_0 + c_1 v_1 + \dots + c_k v_k \qquad (2.4)$$

Geometrically, (2.3) is a k-dim plane that is a translate of the k-dimensional vector subspace

$$Ax = 0 \qquad (2.4)$$

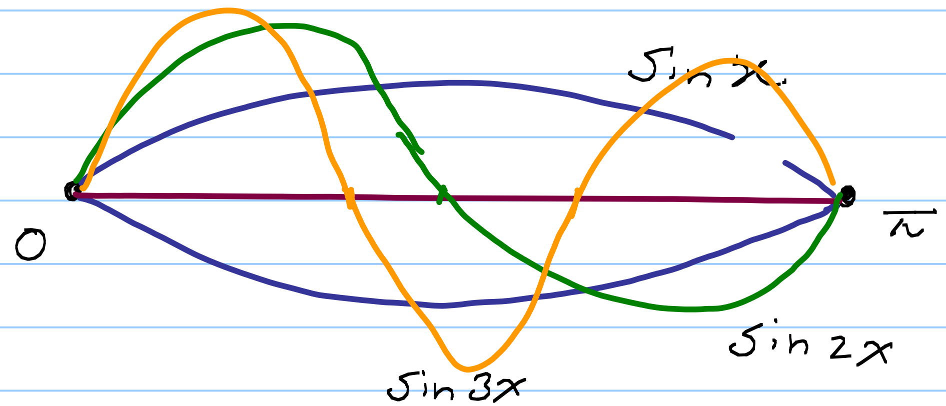
by the vector x_0



Recall from linear algebra the notion of linear independence of a set of vectors.

Likewise one speaks of linear independence of functions. Just as vectors can be subjected to linear operations, one can form "superpositions" of functions (example waves)

The functions may be the modes of vibrations of a (uniform) string clamped at two ends:



Ex: Check that $\sin x, \sin 2x, \ldots, \sin nx$ are linearly independent functions on $[0,\pi]$

This means whenever for constants $c_1, \ldots, c_n$

$$c_1 \sin x + \cdots + c_n \sin nx = 0 \quad \text{for all } x \in [0,\pi], \qquad (2\cdot5)$$

follows $c_1 = c_2 = \cdots = c_n = 0$.

To verify this, multiply (2-5) by $\sin nx$ and integrate over $[0,\pi]$ we get

$$c_n \int_0^\pi \sin^2 nx \, dx = 0 \quad \text{or} \quad c_n = 0.$$

Now multiply by $\sin(n-1)x$ and integrate over $[0, \pi]$

we get $c_{n-1} = 0$. etc;

<u>Example:</u> Check that $e^{m_1 x}, \dots, e^{m_k x}$ are linearly indep. whenever $m_1, \dots, m_k$ are distinct real or complex numbers.

<u>Sol:</u> Suppose there exist constants $c_1, \dots, c_k$ such that

$$c_1 e^{m_1 x} + \dots + c_k e^{m_k x} = 0 \quad \text{--- (2.6)}$$

<u>for all x</u>

Differentiate (2-6) successively w.r.t. x and put $x = 0$ we get

$$
\begin{aligned}
c_1 + c_2 + \cdots &+ c_k = 0 \\
m_1 c_1 + m_2 c_2 + \cdots &+ m_k c_k = 0 \\
m_1^2 c_1 + m_2^2 c_2 + \cdots &+ m_k^2 c_k = 0 \\
\cdots \quad \cdots \quad \cdots & \quad \cdots \quad \cdots \\
m_1^{k-1} c_1 + m_2^{k-1} c_2 + \cdots &+ m_k^{k-1} c_k = 0
\end{aligned}
\tag{2.7}
$$

The determinant of the system is

$$
V = \begin{vmatrix}
1 & 1 & 1 & \cdots & 1 \\
m_1 & m_2 & m_3 & \cdots & m_k \\
m_1^2 & m_2^2 & m_3^2 & \cdots & m_k^2 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
m_1^{k-1} & m_2^{k-1} & m_3^{k-1} & \cdots & m_k^{k-1}
\end{vmatrix}
\tag{2.8}
$$

It is well known (and you are invited to prove) that the value of V (known as the Vandermonde determinant) is

$$V = \prod_{i<j} (m_i - m_j) \neq 0 \qquad (2\text{-}9)$$

since $m_1, \ldots, m_k$ are assumed to be distinct

<u>Ex:</u> Show that $e^{mx}, xe^{mx}, x^2 e^{mx}, \ldots, x^k e^{mx}$ are linearly indep

<u>Ex:</u> Show that $x^k, x^k \ln x, \ldots, x^k (\ln x)^m$ are lin. Indep over $(0, \infty)$

<u>Recall:</u> $$e^{i\theta} = \cos\theta + i\sin\theta \quad (\text{Euler}) \qquad (2.10)$$

The complex exponential

$$e^{(a+ib)x} = e^{ax}(\cos bx + i\sin bx)$$

is a (complex) linear combination of

$$e^{ax}\cos bx \quad \text{and} \quad e^{ax}\sin bx.$$

<u>Theorem:</u> Suppose $y_1(x)$, $y_2(x)$ are two linearly independent solutions of

$$y'' + P(x)y' + Q(x)y = 0 \qquad (2.11)$$

such that

$$y_1(0) = 1,\quad y_1'(0) = 0; \qquad y_2(0) = 0,\quad y_2'(0) = 1.$$

Then, <u>every</u> solution $y(x)$ of (2.11) is of the form

$$c_1 y_1(x) + c_2 y_2(x) \text{ for constants } c_1, c_2$$

<u>In</u> other words the space of solutions of (2.11) is <u>at most</u> a <u>two dimensional</u> vector space (That it is a vector space is clear)

<u>proof:</u> Let $y(x)$ be any solution of (2.11)

Consider

$$z(x) = y(x) - y(0)\, y_1(x) - y'(0)\, y_2(x)$$

Surely $z(x)$ is also a solution. We shall prove $z(x) \equiv 0$

First note

$$z(0) = 0, \qquad z'(0) = 0$$

[Remark

The idea of proof is more popularly known in the context of PDES. It goes under the name of Energy Method.]

Let $\underline{I}$ be the interval on which P, Q are continuous. We have assumed $0 \in \underline{I}$

The initial conditions could have been chosen at any point $x_0 \in \underline{I}$.

Let A be any point in $\underline{I}$

We shall show that $z(A) = 0$

$$M = \textbf{lub}\left\{ |P(x)| + |Q(x)| \,/\, 0 \le x \le A \right\}$$

Define the "<u>Energy function</u>"

$$E(x) = (z(x))^2 + (z'(x))^2 \qquad (2\text{-}12)$$

We show $E(x) \equiv 0$ (Call this the Energy Principle)

$$E'(x) = 2zz' + 2z'z''$$

$$= 2zz' - 2z'(Pz' + Qz)$$

$$= 2zz'(1-Q) - 2z'^2 P \leq |2zz'(1-Q)| + |-2z'^2 P|.$$

$$\leq |2zz'|.(1+M) + 2(z'^2+z^2)M$$

$$\leq (z^2+z'^2)(1+M) + 2M(z^2+z'^2)$$

$$= E(x)(3M+1)$$

We have the linear Differential Inequality:

$E' - (3M+1)E \leq 0$. Multiply by $\exp(-x(3M+1))$:

$$\frac{d}{dx}\left(E(x).\exp -(3M+1)x\right) \leq 0$$

Integrating from 0 to A we get

$$0 \leq E(A) \exp\left(-(3M+1)A\right) \leq E(0) \qquad (2.13)$$

But $E(0) = 0$

$\therefore \quad E(A) = 0$ and the proof is complete

Remark: It is fairly straightforward to extend this result to n-th order Linear ODES

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots + P_n(x)\, y = 0 \qquad (2.14)$$

to show that the solutions of (2.14) forms a vector space of dimension $\leq n$

To show that the dimension is exactly n we need to construct (using Analysis)

n-lin. Indep. Solutions. We shall not do so now.

The Energy Method can be adapted to a variety of situations and is one of the most powerful techniques in the theory of Diff Eqns.

Wronskians: Given two continuously diff. functions $y_1(x)$ and $y_2(x)$, the Wronskian of $y_1(x)$, $y_2(x)$ denoted by $W(y_1, y_2)$ is defined as:

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \qquad (2\cdot 14)$$

For three twice continuously differentiable functions $y_1(x), y_2(x), y_3(x)$, the Wronskian $W(y_1, y_2, w_3)$ is defined as

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Generalization to n-functions is immediate.

Ex: Compute the Wronskian of $e^{m_1 x}, \dots, e^{m_k x}$.

Theorem: If y_1, y_2 are linearly dependent, $W(y_1, y_2)$ vanishes identically.

pf: There exists constants c_1, c_2 such that

$$c_1 y_1 + c_2 y_2 = 0; \quad c_1, c_2 \text{ not both zero}$$

$$\therefore \; c_1 y_1' + c_2 y_2' = 0$$

This is a system of equations for c_1, c_2 with a nontrivial solution. Their determinant must vanish identically.

Unfortunately the Converse is <u>NOT</u> true

<u>Example</u>: Let $f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ x^3 & \text{if } x \leq 0 \end{cases}$

$$g(x) = \begin{cases} x^3 & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Check that $W(f,g) \equiv 0$ but f, g are linearly independent.

To have a result in the opposite direction we need some additional hypothesis.

One such result is the following:

<u>Theorem</u>: If $y_1(x), y_2(x), \ldots, y_n(x)$ are solutions of a linear ODE

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots + P_n(x)\, y = 0 \qquad (2\text{-}15)$$

where $P_1, P_2, \ldots, P_n$ are continuous on an interval I then

$y_1(x), \ldots, y_n(x)$ are linearly indep <u>if</u> <u>and</u> <u>only</u> <u>if</u>

$$W(y_1, \ldots, y_n) \neq 0 .$$

<u>Proof</u>: $W \neq 0 \Rightarrow$ linear indep

(Equivalently lin. dep $\Rightarrow W \equiv 0$)

has already been proved (for $n = 2$ but the proof of the general case is clear and proceeds along similar lines)

Converse: We shall illustrate the result with $n=2$. The general case is similar.

Let $W \equiv 0$. In particular $W(x_0) = 0$. The columns of $W(x_0)$ are linearly dependent

$$c_1 \begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \end{bmatrix} + c_2 \begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \end{bmatrix} = 0 \; ; \quad c_1, c_2 \text{ not both zero}$$

Let $z(x) = c_1 y_1(x) + c_2 y_2(x)$. Then $z(x)$ is a solution of the ODE and

$$z(x_0) = 0 \; ; \quad z'(x_0) = 0.$$ Hence by the **Energy Principle,**

$$z(x) \equiv 0.$$ That is $c_1 y_1(x) + c_2 y_2(x) \equiv 0$, c_1, c_2 not both zero and y_1, y_2 are lin. Dep.

We now obtain an important formula for the Wronskian of n solutions of a homogeneous n-th order linear ODE. This formula shows that either $W \equiv 0$ or W NEVER vanishes.

Lemma (Abel-Liouville formula)

With the set up as in the theorem

$$\frac{dW}{dx} = -P_1(x)W \text{ and so} \qquad (2.18)'$$

$$W(x) = W(x_0) \exp - \int_{x_0}^{x} p(t)\,dt \qquad (2.18)''$$

Proof: We illustrate the proof for $n = 3$.

$$\frac{dW}{dx} = \frac{d}{dx}\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}$$

$$= -\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ P_1 y_1'' + P_2 y_1' + P_3 y_1 & P_1 y_2'' + P_2 y_2' + P_3 y_2 & P_1 y_3'' + \cdots \\ & & \cdots \end{vmatrix}$$

$$= -P_1(x)\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = -P_1(x)\, W$$

This proves (2-18)'. The next formula follows immediately upon mult. by $\exp - \int_{x_0}^{x} p_1(t)\,dt$

and integrating. (N.B: Avoid dividing by W)

Application of Abel Liouville Formula

Theorem: Suppose $y_1(x)$ is a solution of

$$y'' + P(x)\,y' + Q(x)\,y = 0 \qquad (2\text{-}19)$$

then a second linearly independent solution is given by

$$y_2(x) = y_1(x) \int \frac{1}{y_1(x)^2}\left[\exp - \int P(x)\,dx\right] dx$$

Proof:

$$W = W(y_1, y_2) = y_1 y_2' - y_1' y_2 \qquad (2\text{-}20)$$

Now W is known via Abel Liouville formula and so (2-20) is a linear first order ODE for $y_2(x)$:

$$y_2'(x) - \frac{y_1'(x)}{y_1(x)} y_2(x) = \frac{W}{y_1(x)} = \frac{\exp - \int p(x)dx}{y_1(x)}$$

Integrating factor for this equation is

$$\exp \int \left(-y_1'/y_1\right) = \frac{1}{y_1(x)}$$

So applying the formula for the solution of a linear first-order ODE

$$\frac{y_2(x)}{y_1(x)} = \int \frac{\left[\exp - \int p(x)dx\right]}{y_1(x)^2} dx$$

$$y_2(x) = y_1(x) \int \frac{\exp\left(-\int p(x)\,dx\right)}{(y_1(x))^2}\,dx$$

Proof is complete.

Example: (1) $(1-x^2)y'' - 2xy' + 2y = 0$

Note that $y_1(x) = x$ is one solution

By the formula obtained above

Important: $P(x) = \dfrac{-2x}{1-x^2}$ (!)

$$y_2(x) = x \int \frac{1}{x^2}\left(\exp \int \frac{-2x}{1-x^2}\,dx\right)dx$$

(2) Show that $3x - 4x^3$ is one solution of

$$(1-x^2)y'' - xy' + 9y = 0$$

Determine the second solution.

(3) Work out problems in Q8 (Tut Sheet 4) on page 8

(4) Solve $$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = 0$$

given that $y_1(x) = \frac{\sin x}{\sqrt{x}}$ is one solution

Q7 on page 8: Three solutions of a certain **second** order **non-homogeneous** ODE are

$$y_1(x) = 1 + e^{x^2}, \quad y_2(x) = 1 + xe^{x^2}, \quad y_3(x) = (1+x)e^{x^2} - 1$$

Find the general solution.

Note that
$$y_1(x) - y_2(x) = e^{x^2}(1-x)$$
$$y_3(x) - y_2(x) = 2 + e^{x^2}$$
are solutions of the homogeneous equation.

Moreover these are linearly independent (Check)

Hence $A\left(e^{x^2}(1-x)\right) + B\left(2 + e^{x^2}\right)$ is the general solution of the associated homogeneous equation.

Pick $y_1(x)$ as a particular solution.

The general solution of the inhomogeneous eq^n is

$$1+e^{x^2}+A\left(e^{x^2}(1-x)\right)+B\left(2+e^{x^2}\right)$$

Q14 on p.8: To find a homogeneous linear ODE whose general solution is

$$c_1 x^2 e^x + c_2 x^3 e^x.$$

Since the general solution has two arbitrary constants the order of the DE must be 2.

The required ODE is $\begin{vmatrix} y & x^2e^x & x^3e^x \\ y' & (x^2e^x)' & (x^3e^x)' \\ y'' & (x^2e^x)'' & (x^3e^x)'' \end{vmatrix} = 0$

Expand and write out the ODE.

Examples: Find the complete solution of the following one solution is given

(i) $x^2y'' + xy' - y = 0 \quad (y_1(x) = x)$

(ii) $x(x\cos x - 2\sin x)y'' + (x^2+2)\sin x\, y' - 2(x\sin x + \cos x)y = 0 \quad (y_1(x) = x^2)$

(iii) $x^2y'' + xy' - 9y = 0 \quad (y_1(x) = x^3)$

(iv) $(x+2)y'' - (2x+5)y' + 2y = 0 \quad (y_1(x) = e^{2x})$

END OF CHAPTER 2.

Chapter III

Differential Equations with Const. Coefficients

The Method of undetermined Coefficients.

We consider ODEs of the form

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = Q(x) \quad (3\cdot 1)$$

where $Q(x)$ is of the form

$$Q_0(x) + Q_1(x) e^{m_1 x} + \cdots + Q_k(x) e^{m_k x}$$

m_k may be real or complex constants.

$Q_j(x)$ are polynomials in x

Step I: Find a complete solution of the associated homogeneous equation

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0 \qquad (3.2)$$

This complete solution involving n arb. constants is called the **Complimentary function. (CF)**

Step II: Find one solution of the inhomogeneous equation (3.1) called the **Particular integral. (PI)**

Step III: The general solution of (3.1) is then

$$y(x) = CF + P.I \qquad (3.3)$$

I) Finding the Complimentary Function

Substitute the trial solution e^{mx} in (3.2) and we get the polynomial Equation:

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0 \qquad (3\text{-}2)$$

$$m^n + a_1 m^{n-1} + \cdots + a_n = 0 \qquad (3\text{-}3)$$

Remark: Equation (3.3) is called the characteristic Equation. It is infact the Char. Equation of a certain matrix associated with (3-2). However we shall not need this fact and so we shall say no more on this.

If $m_1, \dots, m_k$ are distinct roots of (3.3) (Char. Roots) then $e^{m_1 x}, \dots, e^{m_k x}$ (3.4) is a list of k - linearly independent solutions of (3.2). Thus, if all roots of the characteristic equation are <u>distinct</u> the problem of finding the CF is over! It is simply

$$c_1 e^{m_1 x} + \cdots + c_n e^{m_n x} \qquad (3.5)$$

For example: (i) $c_1 e^x + c_2 e^{-x}$ is the CF

for $y'' - y = 0$

(ii) $c_1 e^{2x} + c_2 e^{-x}$ is the CF for

$$y'' - y' - 2y = 0$$

We now turn to the general case

Case of Repeated Roots:

Let us for simplicity write

$$Ly = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$$

and regard L as an "Operator" that acts on a smooth function y to produce another smooth function Ly. Then

$$L e^{mx} = (m^n + a_1 m^{n-1} + \cdots + a_n) e^{mx}$$

i.e $$L(e^{mx}) = p(m) e^{mx} \qquad (3\cdot 6)$$

Where $p(m)$ denotes the characteristic polynomial

RHS of $(3\cdot 6)$ is zero iff $m = m_0$ is a root of $p(m)$. That is to say iff $p(m_0) = 0$

Now suppose that m_0 is a double root

Then

$$P(m_0) = 0 = P'(m_0)$$

Let us diff the identity

$$L(e^{mx}) = P(m)\, e^{mx}$$

w.r.t m

$$\frac{\partial}{\partial m} L(e^{mx}) = \frac{\partial}{\partial m}\left(P(m)\, e^{mx}\right)$$

$$\therefore\ L\left(\frac{\partial}{\partial m} e^{mx}\right) = P'(m)\, e^{mx} + P(m)\, e^{mx} \cdot x$$

$$\therefore\ L(x e^{mx}) = P'(m)\, e^{mx} + P(m)\, x e^{mx} \qquad (3.7)$$

If now $P(m_0) = P'(m_0) = 0$ we see from (3.7) that

$$L(xe^{mx}) = 0$$

In otherwords besides e^{mx}, xe^{mx} is also a solution of the ODE (3.2)

It is clear how we must proceed in general. For example if m_0 is a **TRIPLE ROOT**

then $$P(m_0) = P'(m_0) = P''(m_0) = 0.$$

We diff. Eqn (3.7) <u>again</u> w.r.t m to get

$$L(x^2 e^{mx}) = (P''(m) + 2P'(m)x + P(m).x^2)e^{mx} \quad (3.8)$$

Put $m = m_0$ and we get

$$L\left(x^2 e^{mx}\right) = 0 \tag{3-9}$$

We see that $e^{mx}, \; x e^{mx}, \; x^2 e^{mx}$ (3-10)
are three lin. Indep. Solutions.

<u>Theorem</u>: If the Characteristic Equation has roots $m_1, \ldots, m_k$ of multiplicities $\mu_1, \ldots, \mu_k$ then

$$\begin{aligned}
&e^{m_1 x}, \; x e^{m_1 x}, \; \ldots, \; x^{\mu_1 - 1} e^{m_1 x}; \\
&e^{m_2 x}, \; x e^{m_2 x}, \; \ldots, \; x^{\mu_2 - 1} e^{m_2 x}; \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots; \\
&e^{m_k x}, \; x e^{m_k x}, \; \ldots, \; x^{\mu_k - 1} e^{m_k x}
\end{aligned} \tag{3-11}$$

is a complete list of n - linearly independent solutions of the equation

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0 \quad (3\cdot 2)$$

REMARK: Case of Complex Roots is Subsumed in Our discussion.

If $m = a + ib$ is a Complex triple root

Then $a - ib$ is <u>also</u> a Complex triple root

(Our Coefficients $a_1, \dots, a_n$ in (3-2) are assumed **REAL**)

Corresponding to these six roots

$a+ib,\ a+ib,\ a+ib;\ a-ib,\ a-ib,\ a-ib$

We have the solutions

$$e^{ax}(\cos bx \pm i\sin bx)$$

$$x\,e^{ax}(\cos bx \pm i\sin bx)$$

$x^2e^{ax}(\cos bx \pm i\sin bx)$. Seperating the real and imaginary parts we have the six linearly independent solutions

$e^{ax}\cos bx,\quad x\,e^{ax}\cos bx,\quad x^2e^{ax}\cos bx$

$e^{ax}\sin bx,\quad x\,e^{ax}\sin bx;\quad x^2e^{ax}\sin bx$

__Example:__ Solve $y^{(4)} + y = 0$ (3.12)

The Char. poly is $m^4 + 1 = 0$

__Roots:__ $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

$\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + i/\sqrt{2}$

$\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{-i}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

__Sol:__ $e^{x/\sqrt{2}}\left(c_1\cos\frac{x}{\sqrt{2}} + c_2\sin\frac{x}{\sqrt{2}}\right) +$ (3.13)

$e^{-x/\sqrt{2}}\left(c_3\cos x/\sqrt{2} + c_4\sin\frac{x}{\sqrt{2}}\right)$

Exercise: Solve $y^{(4)} + y^{(2)} + y = 0$

Finding the Particular Integral:

Method of undetermined coefficients.
Suppose we wish to solve

$$y'' + y = e^x. \tag{3-14}$$

$$CF = c_1 \cos x + c_2 \sin x.$$

In what follows $D = \frac{d}{dx}$ (3.15)

$$De^x = e^x \; ; \; D^2 e^x = e^x \text{ so}$$
$$(D^2+1)e^x = D^2 e^x + e^x = 2e^x$$

and hence

$$(D^2+1)\left(\frac{1}{2}e^x\right) = e^x$$

This shows that $\frac{1}{2}e^x$ is a particular solution of (3-14)

The complete solution is then

$$c_1 \cos x + c_2 \sin x + \frac{1}{2}e^x \qquad (3\text{-}16)$$

More generally if we had instead

$$y'' + y = e^{ax}$$

then we ought to try and find a P.I. in the form Ae^{ax} where the coefficient A is to be suitably chosen.

As yet A is an "Undetermined Coefficient"

Substituting Ae^{ax} in the ODE

$$(D^2+1)y = y'' + y = e^{ax}$$

$$(D^2+1)Ae^{ax} = e^{ax}$$

$$\therefore (a^2+1)A = 1$$

$$\therefore A = \frac{1}{1+a^2}$$

Whereby we conclude $PI = \dfrac{e^{ax}}{1+a^2}$

The Complete Solution is

$$c_1 \cos x + c_2 \sin x + e^{ax}/(1+a^2)$$

We tacitly assumed that a was real!
The method clearly works even for <u>complex</u> a
Except $a = \pm i$ (!)

Consider now the ODE

$$y'' - y = e^{-x} \qquad (3.17)$$

Substituting Ae^{-x} would lead to the relation

$$0 = e^{-x} \; !$$

Note that

$$(D+1)\, e^{-x} = 0 .$$

Let y_p be a particular integral of (3.17)
That is to say

$$(D^2 - 1)\, y_p = e^{-x} \qquad (3\text{-}18)$$

Apply $(D+1)$ to both sides:

$$(D+1)(D^2-1)\, y_p = 0 \qquad (3\text{-}19)$$

Thus, y_p satisfies a third order HOMOGENEOUS equation with const. coeff. Hence y_p must be sought among the general sol^n of (3-19)

That is

$$y_p = c_1 e^{x} + c_2 e^{-x} + \underline{\underline{A}}\, x e^{-x} \qquad (3\text{-}20)$$

We substitute (3-20) into (3-18) to get $\underline{\underline{A}}$.

Note that when this is executed, the terms in (3-20) with coefficients c_1, c_2 would <u>DROP</u>

OUT. Moreover $c_1 e^{x} + c_2 e^{-x}$ is ALREADY a part of the Complimentary functions.

Hence

$$y_p = \cancel{c_1 e^{x}} + \cancel{c_2 e^{-x}} + A\,x e^{-x}$$

$$= A\,x e^{-x}. \qquad (3.21)$$

is the Correct form of y_p.

We substitute (3.21) in the ODE (repeated here for convenience)

$$(D^2 - 1)\, y_p = e^{-x} \qquad (3.18)$$

Let us execute this systematically:

$$(D+1)(Axe^{-x}) = A\left(D(xe^{-x}) + xe^{-x}\right)$$

$$= A\left(e^{-x} - \cancel{xe^{-x}} + \cancel{xe^{-x}}\right)$$

$$= Ae^{-x}$$

$$(D^2-1)(Axe^{-x}) = (D-1)(Ae^{-x})$$

$$= A(-2e^{-x})$$

Hence $A(-2e^{-x}) = e^{-x}$

or $A = \frac{-1}{2}$

$\therefore \quad y_p = \frac{-x}{2} e^{-x}$; Gen. Sol: $c_1 e^{x} + c_2 e^{-x} - \frac{1}{2} e^{-x} x$

Annihilator: The idea is that to find a y_p satisfying

$$P(D)\, y_p = Q(x) \qquad (3\text{-}22)$$

(Where $Q(x)$ is a sum of terms of the form $c\,x^k e^{mx}$)

One finds a polynomial $N(D)$ that annihilates $Q(x)$: $\quad N(D)\,Q(x) = 0$

So applying $N(D)$ to both sides of (3-22)

$$N(D)\,P(D)\, y_p = 0$$

thereby y_p is a solution of a <u>homogeneous</u> equation with constant coeff. which we can handle

So in the example $y'' - y = e^{-x}$

$(D+1)$ was the annihilator for e^{-x}

We make a list of annihilators:

Function	Annihilator
e^{ax}	$(D-a)$
$x e^{ax}$	$(D-a)^2$
$x^2 e^{ax}$	$(D-a)^3$
	etc;

Now suppose RHS is $e^{-x} + xe^{x}$

For example solve $y'' - y = 3e^{-x} + xe^{x}$

$(D+1)$ annihilates $3e^{-x}$

$(D-1)^2$ annihilates xe^{x}

The product $(D+1)(D-1)^2$ annihilates the sum $3e^{-x} + xe^{x}$

What is the annihilator of

$3e^{-x} + 2x^2e^{-x}$?

Ans: $(D+1)^3$

Rule: Suppose $N_1(D), \dots, N_k(D)$ are annihilators of $Q_1(x), \dots, Q_k(x)$ then The sum $Q_1(x) + \dots + Q_k(x)$ is annihilated by the L.C.M of

$$N_1(D), \dots, N_k(D).$$

Let us now determine the annihilator of

$$x \sin x$$

Sol: $$x \sin x = \frac{1}{2i} x e^{ix} - \frac{1}{2i} x e^{-ix}$$

$(D-i)^2$ annihilates xe^{ix}

$(D+i)^2$ annihilates xe^{-ix}

Hence $(D-i)^2(D+i)^2$ or $(D^2+1)^2$

annihilates $x\sin x$

__Exercise:__ Check that

$((D-2)^2+1)^2$ is the annihilator

for $xe^{2x}\sin x$ and $xe^{2x}\cos x$.

We can now complete the table of annihilators.

Function	Annihilator	
e^{ax}	$(D-a)$	
$x\,e^{ax}$	$(D-a)^2$	(I)
$x^2\,e^{ax}$	$(D-a)^3$	
... ...	... ...	
$\sin bx$; $\cos bx$	D^2+b^2	
$x\sin bx$; $x\cos bx$	$(D^2+b^2)^2$	(II)
$x^2\sin bx$, $x^2\cos bx$	$(D^2+b^2)^3$	
... ...	... .	
$e^{ax}\cos bx$; $e^{ax}\sin bx$	$(D-a)^2+b^2$	(III)
$xe^{ax}\cos bx$; $xe^{ax}\sin bx$	$((D-a)^2+b^2)^2$	
$x^2e^{ax}\cos bx$; $x^2e^{ax}\sin bx$	$((D-a)^2+b^2)^3$	
... ...	... ...	

<u>N.B:</u> Observe that if u and v are particular solutions of

$$P(D)y = Q_1(x)$$
$$P(D)y = Q_2(x)$$

then $u+v$ is a particular solution of

$$P(D)y = Q_1(x) + Q_2(x)$$

It may be computationally easier to break up the problem and solve

$$\left.\begin{array}{l} P(D)y = Q_1(x) \\ P(D)y = Q_2(x) \end{array}\right\} \text{ separately.}$$

Example. Solve $y'' + y = \sin x$

CF : $c_1 \cos x + c_2 \sin x$

(D^2+1) annihilates RHS. Let y_p be a particular integral.

$$(D^2+1)\, y_p = \sin x \qquad (3.23)$$

$$\therefore\ (D^2+1)^2\, y_p = 0$$

$$\therefore\ y_p = \cancel{c_1 \cos x} + \cancel{c_2 \sin x} + A x \cos x + B x \sin x$$

We substitute this formulation (Ansatz) into the ODE (3.23):

$$A(D^2+1)\,x\cos x + B(D^2+1)(x\sin x) = \;?$$

$$D^2(x\cos x) = -x\cos x - 2\sin x$$

$$(D^2+1)(x\cos x) = -2\sin x$$

Likewise $D^2(x\sin x) = -x\sin x + 2\cos x$

$$\therefore (D^2+1)(x\sin x) = 2\cos x$$

$$\therefore -2A\sin x + 2B\cos x = \sin x$$

$$\therefore B = 0; \quad A = -\tfrac{1}{2}$$

$$y_p = -\frac{1}{2}\,x\cos x$$

$\boxed{\text{Gen. Sol}^n\text{:} \quad c_1\cos x + c_2\sin x - \frac{1}{2}x\cos x}$

Solve: $y'' - 4y = x^2 e^{2x} + e^{3x}$

<u>CF</u>: $c_1 e^{2x} + c_2 e^{-2x}$

A solution of $y'' - 4y = e^{3x}$ is easily found to be $\frac{1}{5} e^{3x}$ (Check)

We now find a particular solution of

$$y'' - 4y = x^2 e^{2x}$$

i.e $(D^2 - 4)\, y_p = x^2 e^{2x}$

$\therefore (D-2)^4 (D+2)\, y_p = 0$

$$\therefore y_p = \cancel{c_1 e^{2x}} + \cancel{c_2 e^{-2x}} + Axe^{2x} + Bx^2 e^{2x} + Cx^3 e^{2x} \quad (3.24)$$

Substituting the Ansatz (3.24) into the ODE

$$(D^2-4)(Axe^{2x}+Bx^2e^{2x}+Cx^3e^{2x})$$

$$= x^2e^{2x}$$

First apply $(D-2)$:

$$(D-2)(xe^{2x}) = e^{2x}$$
$$(D-2)(x^2e^{2x}) = 2x\,e^{2x}$$
$$(D-2)(x^3e^{2x}) = 3x^2e^{2x}$$

$$\therefore (D^2-4)(xe^{2x}) = \cancel{(D-2)}e^{2x} + 4e^{2x} = 4e^{2x}$$

$$(D^2-4)(x^2e^{2x}) = (D-2)(2xe^{2x}) + 8xe^{2x}$$

$$= 2e^{2x} + 8xe^{2x}$$

$$(D^2-4)(x^3e^{2x}) = (D-2)(3x^2e^{2x}) + 12x^2e^{2x}$$

$$= 6xe^{2x} + 12x^2e^{2x}$$

We get finally

$$4Ae^{2x} + B(2e^{2x} + 8xe^{2x}) + C(6xe^{2x} + 12x^2e^{2x}) = x^2e^{2x}$$

$$\therefore \quad 12C = 1 \quad \text{or} \quad C = 1/12$$

$$8B + 6C = 0 \quad \text{or} \quad B = -1/16$$

$$4A + 2B = 0 \quad \text{or} \quad A = 1/32$$

$$\tilde{y}_p = \frac{1}{32}xe^{2x} - \frac{1}{16}x^2e^{2x} + \frac{1}{12}x^3e^{2x}$$

<u>gen. sol:</u> $\tilde{y}_p + c_1e^{2x} + c_2e^{-2x} + \frac{1}{5}e^{3x}.$

Q10. (iv)

$$y^{(4)} + y = x e^{x/\sqrt{2}} \sin\left(\frac{x}{\sqrt{2}}\right)$$

Annihilator for $x e^{x/\sqrt{2}} . \exp\left(\frac{ix}{\sqrt{2}}\right)$

is $\left(D - \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right)^2$

Annihilator for $x e^{x/\sqrt{2}} \exp\left(-\frac{ix}{\sqrt{2}}\right)$ is

$$\left(D - \left(\frac{1}{\sqrt{2}} - i/\sqrt{2}\right)\right)^2$$

$$\sin\left(\frac{x}{\sqrt{2}}\right) . x e^{x/\sqrt{2}} = \frac{1}{2i}\Big(x e^{x/\sqrt{2}} e^{ix/\sqrt{2}} - x e^{x/\sqrt{2}} e^{-ix/\sqrt{2}} \Big)$$

So annihilator for $x\left(\sin \frac{x}{\sqrt{2}}\right) e^{x/\sqrt{2}}$ is the LCM:

$$\left(\left(D - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\right)^2 = \left(D^2 - \sqrt{2}\,D + 1\right)^2$$

<u>Note:</u> $m^4 + 1 = \left(\left(m - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\right)\left(\left(m + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}\right)$

So equation for y_p is

$$\left(D^2 - \sqrt{2}\,D + 1\right)^3 \left(D^2 + \sqrt{2}\,D + 1\right) y_p = 0$$

$$y_p = e^{x/\sqrt{2}} \cos \frac{x}{\sqrt{2}} \left(Ax + Bx^2\right)$$

$$+ e^{x/\sqrt{2}} \sin \frac{x}{\sqrt{2}} \left(Cx + Dx^2\right)$$

Cauchy - Euler Equation.

These are equations of the form

$$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \dots + a_{n-1} x y' + a_n y = 0 \quad (3.25)$$

E.g.:

$$x^2 y'' - y = 0$$

$$x^2 y'' + x y' + y = 0$$

To solve the Cauchy Euler Equation take $y = x^m$ as a TRIAL Solution

Substituting in (3.25) we get the polynomial equation for m:

$$m(m-1)\cdots(m-n+1) + a_1\, m(m-1)\cdots(m-(n-2)) + \cdots + a_{n-1}\, m + a_n = 0 \quad (3.26)$$

Equation (3.26) is called the **Indicial Equation**

If $m_1, \ldots, m_k$ are **DISTINCT** roots of the Indicial Equation then

$$x^{m_1}; \ldots; x^{m_k}$$ are linearly Indep. Solutions of the ODE.

Hence, if the Indicial Eq^n has n-distinct roots $m_1, \ldots, m_n$ then

$C_1 x^{m_1} + \cdots + C_n x^{m_n}$ is the complete solution of (3.25).

Case of Equal Roots.

Suppose the indicial Polynomial is $I(m)$

Denote

$$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_n y$$

as Ly

Then $L x^m = I(m) x^m \qquad (3.27)$

Diff. both sides with respect to m:

$$\frac{\partial}{\partial m} L(x^m) = I'(m)x^m + I(m)\, x^m \ln x$$

$$\therefore L\left(\frac{\partial}{\partial m} x^m\right) = I'(m)x^m + I(m)\, x^m \ln x$$

$$\therefore L(x^m \ln x) = x^m \left(I'(m) + I(m) \ln x\right) \qquad (3.28)$$

If $m = m_0$ is a double root of $I(m)$ then

$$I(m_0) = I'(m_0) = 0 \text{ so } (3.28) \text{ gives}$$

$x^{m_0} \ln x$ is also a solution

Likewise if m_0 is a triple root then

x^{m_0}, $x^{m_0} \ln x$; $x^{m_0} (\ln x)^2$ are three

Solutions of the ODE corresponding to the index m_0.

Complex Roots: $m = a + ib$ say

$$x^m = \exp m \ln x \qquad \text{(Definition)}$$

$$= \exp(a \ln x + ib \ln x)$$

$$= (\exp a \ln x) \cdot (\cos b \ln x + i \sin b \ln x)$$

$$= x^a (\cos b \ln x + i \sin b \ln x)$$

The real solutions are

$$x^a \cos b \ln x \quad \text{and} \quad x^a \sin b \ln x$$

If $a+ib$ is a double root then so is $a-ib$.

The four solutions are

$$x^a \cos b \ln x \; ; \quad x^a \sin b \ln x$$

$$x^a \ln x \cdot \cos b \ln x \; ; \quad x^a \ln x \sin b \ln x$$

It is clear how the general case must proceed.

Inhomogeneous Cauchy - Euler equations

Q12 page 8 Tut Sheet 4.

$$x^2 y'' + 2xy' - 6y = \begin{cases} 10x^2 \\ 10x^4 \end{cases} \qquad (3.29)$$

$$I(m) = m(m-1) + 2m - 6 = m^2 + m - 6$$
$$= (m+3)(m-2)$$

$c_1 x^{-3} + c_2 x^2$ is the **C.F.**

To find the particular integral find an annihilator for RHS $10x^2$.

Note that D^3 does it but if we apply D^3 to (3.29) we <u>do</u> <u>not</u> <u>get</u> a Cauchy Euler equation

Note that $\left(x\frac{d}{dx} - 2\right)$ annihilates $10x^2$

Apply $\left(x\frac{d}{dx} - 2\right)$ to both sides of (3.29)

$$\left(x\frac{d}{dx} - 2\right)\left(x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} - 6\right)y_p = 0 \qquad (3.30)$$

Now
$$x^2\frac{d^2}{dx^2} = x\frac{d}{dx}\left(x\frac{d}{dx} - 1\right)$$

$$x^3\frac{d^3}{dx^3} = \left(x\frac{d}{dx}\right)\left(x\frac{d}{dx} - 1\right)\left(x\frac{d}{dx} - 2\right)$$

etc;

Equation (3.30) can be more conveniently written as

$$\left(x\frac{d}{dx}-2\right)\left(\left(x\frac{d}{dx}\right)^2+x\frac{d}{dx}-6\right)y_p=0$$

$$\left(x\frac{d}{dx}-2\right)\left(x\frac{d}{dx}-2\right)\left(x\frac{d}{dx}+3\right)y_p=0$$

$$y_p=\cancel{c_1}x^{-3}+\cancel{c_2}x^2+\underline{\underline{A}}\,x^2\ln x$$

A is an undetermined coefficient.

Substituting $y_p = A(x^2\ln x)$ into the ODE we find A.

$$\left(x\frac{d}{dx}\right)\left(Ax^2\ln x\right)$$

$$= 2Ax^2\ln x + Ax^2$$

$$\therefore \left(x\frac{d}{dx}-2\right)\left(Ax^2\ln x\right) = Ax^2$$

$$\left(x\frac{d}{dx}+3\right)\left(x\frac{d}{dx}-2\right)\left(Ax^2\ln x\right) =$$

$$\left(x\frac{d}{dx}+3\right)\left(Ax^2\right) = 5Ax^2$$

$$\therefore \quad 5Ax^2 = 10x^2$$

$$\therefore \quad \underline{\underline{A = 2}}$$

$y_p = 2x^2 \ln x$ and the Complete Solution is

$$\boxed{c_1 x^{-3} + c_2 x^2 + 2x^2 \ln x}$$

Do Exercise 11 on page 8 and Ex 12. page 8 of Tut Sheet 4.

End of Chapter III.

IV - The Method of Variation of Parameters.

This is a powerful method to find the particular Integral when the Complimentary function (the Complete Solution of the Associated homogeneous equation) is Known. The method was **developed by Lagrange** in Connection with problems in Celestial Mechanics.

The method Can be adapted to non-linear ODEs as well. Indeed Lagrange

Originally applied his method to non linear systems.

In the context of PDEs this method is known under the name of **Duhamel's Principle.**

The method can be applied to obtain certain Representation formulas for solutions of linear DEs – Scattering theory in Quantum Mechanics.

The Method We illustrate the method for a second order ODE

$$y'' + P(x)\,y' + Q(x)\,y = R(x) \qquad (4.1)$$

Assume that the complete solution of the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0 \qquad (4.2)$$

is known.

Let $y_1(x)$, $y_2(x)$ be two linearly Indep. Solutions of (4.2).

We seek $y_p(x)$ (Particular Solution of (4.1)) in the form

$$y_p(x) = v_1(x)\, y_1(x) + v_2(x)\, y_2(x) \qquad (4.3)$$

(Assumption I)

$$y_p'(x) = (v_1' y_1 + v_2' y_2) + (v_1 y_1' + v_2 y_2')$$

Now place the restriction

$$v_1' y_1 + v_2' y_2 = 0 \qquad (4.4)$$

(Assumption – II)

Then

$$y_p' = v_1 y_1' + v_2 y_2'$$

$$\therefore \quad y_p'' = v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2'$$

Substituting these into the ODE (4.1) gives

$$\boxed{\begin{aligned} & v_1 y_1'' + v_2 y_2'' \\ & P(x)(v_1 y_1' + v_2 y_2') \\ & Q(x)(v_1 y_1 + v_2 y_2) \end{aligned}} \begin{aligned} & + (v_1' y_1' + v_2' y_2') \\ & + \\ & = R(x) \end{aligned}$$

Now $v_1 y_1'' + P(x) v_1 y_1' + Q(x) v_1 y_1 = 0$

$v_2 y_2'' + P(x) v_2 y_2' + Q(x) v_2 y_2 = 0$

and we are left with

$$v_1' y_1' + v_2' y_2' = R(x) \qquad (4.5)$$

(4.4) and (4.5) together give us the pair of equations

$$\left.\begin{aligned} v_1' y_1 + v_2' y_2 &= 0 \\ v_1' y_1' + v_2' y_2' &= R(x) \end{aligned}\right\} \quad (4.6)$$

This pair can be solved <u>uniquely</u> for v_1', v_2'

because $\{y_1, y_2\}$ are lin. Indep and so the determinant of the system (4.6) which is the Wronskian $\neq 0$

All that remains now is to solve (4.6) for v_1', v_2'

$$\left.\begin{aligned} v_1' &= \text{-----} \\ v_2' &= \text{-----} \end{aligned}\right\} \qquad (4.7)$$

Integrate and obtain v_1, v_2 and then recall:

$$y_p = v_1 y_1 + v_2 y_2$$

Voila (!)

The method works !!

Let us illustrate the method by means of the familiar example

$$y'' + y = \sin x$$

$$y_1(x) = \sin x, \quad y_2(x) = \cos x$$

$$y_p(x) = (\sin x)\, v_1(x) + \cos x \; v_2(x) \qquad (4.8)$$

The pair of Equations for v_1, v_2 are

$$\left.\begin{aligned} v_1' \sin x + v_2' \cos x &= 0 \\ v_1' \cos x - v_2' \sin x &= \sin x \end{aligned}\right\} \quad (4.9)$$

Solving (4-9) we get

$$v_1' = \sin x \cos x \; ; \quad v_2' = -\sin^2 x$$

$$v_1 = -\frac{1}{4}(\cos 2x) \; ; \quad v_2 = \frac{1}{4}\sin 2x - \frac{x}{2}$$

$$\therefore y_p(x) = \left[-\frac{1}{4}\cos 2x \sin x + \frac{1}{4}\sin 2x \cos x \right] - \frac{x}{2}\cos x$$

<u>Note:</u> This part is $\frac{1}{4}\sin x$ which is a part of the Complimentary function.

Ex: Solve $y'' + y = \tan x$

$$y_1(x) = \sin x, \quad y_2(x) = \cos x.$$

$$y_p(x) = (\sin x)\, v_1(x) + (\cos x)\, v_2(x)$$

$$v_1'(x) \sin x + v_2'(x) \cos x = 0$$

$$v_1'(x) \cos x - v_2'(x) \sin x = \tan x$$

$$\therefore \quad v_1'(x) = \tan x \cos x = \sin x$$

or $\quad v_1(x) = -\cos x$

$$v_2'(x) = -\tan x \sin x$$

$$\therefore \quad v_2(x) = \sin x - \log(\sec x + \tan x)$$

So $\quad y_p(x) = -\cos x . \sin x$

$$+ \cos x \,(\sin x - \log(\sec x + \tan x))$$

$$= -\cos x \log(\sec x + \tan x)$$

Physical/geometrical interpretation of the Lagrangian Ansatz.

The idea of taking $y_p(x)$ in the form

$$y_1(x)\, v_1(x) + y_2(x)\, v_2(x)$$

is motivated by Celestial Mechanics. Infact the method was applied by Lagrange, to a non linear system – **the perturbed two body problem Case of a Comet moving in elliptical Orbit.**

The system can be written Aymbolically as

$$L[y] = R(x) \qquad (4.10)$$

where $R(x)$ is a mild <u>periodic</u> disturbing function.

The unperturbed motion is governed by

$$L[y] = 0 \qquad (4.11)$$

Equation (4.11) is Completely Solvable with 5 arbitrary Constants C_1, C_2, C_3, C_4 and C_5

(i) Major axis
(ii) Eccentricity
(iii) Inclination of the major axis with a fixed reference line in the plane of motion
(iv) Direction Cosines of the normal to this plane.

The solution of (4·11) (unperturbed motion) is a **fixed ellipse** with parametric Equation

$$\vec{y} = \vec{y}(x, c_1, c_2, c_3, c_4, c_5) \qquad (4{\cdot}12)$$

(x is the time variable)

The mild periodic forcing function causes a precession of the ellipse (4·12). The "ellipse itself moves" or "rotates" very slowly. This slow precession of the ellipse is superimposed on the (relatively) fast motion along the "ellipse".

Thus c_1, c_2, c_3, c_4, c_5 would no longer be constants but would be **slowly varying functions** of the time variable x leading to the Ansatz

$$y = y(x,\ c_1(x),\ c_2(x), \ldots,\ c_5(x)) \tag{4.13}$$

for the Solution.

In the case of a linear ODE, c_1, c_2, c_3, c_4, c_5 appear linearly in (4.12) and Correspondingly the Ansatz (4.13) assumes the form

$$y_p(x) = c_1(x)\, y_1(x) + \ldots + c_5(x)\, y_5(x)$$

A case in point is the residual precession in Planet Mercury's Orbit. This residual precession is $43''$ per century

Observed by astronomers of 18th Century.

This was explained by Einstein – One of the triumphs of General Th. of Relativity.

The Relativistic Correction term to the two body problem acts like a mild forcing function to the undisturbed Newtonian two body problem.

This Causes a 43″(per cent) precession of Mercury's Perihelion, super-imposed on the (fast) revolution of the planet around the Sun (58 days is the period of revolution which is very fast compared to the 43" per cent. motion).

Returning now to 2nd order ODE

$$y'' + P(x)y' + Q(x)y = R(x) \qquad (4.14)$$

with $y_1(x)$ and $y_2(x)$ as lin. independent solutions of

$$y'' + P(x)y' + Q(x)y = 0 \qquad (4.15)$$

Assume that at time $x = \xi$ the forcing function $R(x)$ is **Shut off**

Let $Y(x, \xi)$ denote the modified solution for $x \geq \xi$

Then $$Y(x,\xi) = c_1(\xi)\, y_1(x) + c_2(\xi)\, y_2(x)$$

Note that the original solution $y(x)$ and the "fictitious solution" $Y(x,\xi)$ would be tangential at $x=\xi$.

Hence

$$\left.\begin{aligned} Y(x,\xi) &= y(x) \\ \frac{d}{dx} Y(x,\xi) &= \frac{d}{dx} y(x) \end{aligned}\right\} \text{ at } x=\xi$$

from which follows:

$$c_1(\xi)\, y_1(\xi) + c_2(\xi)\, y_2(\xi) = y(\xi) \quad (4.16)$$

$$c_1(\xi)\, y_1'(\xi) + c_2(\xi)\, y_2'(\xi) = y'(\xi) \quad (4.17)$$

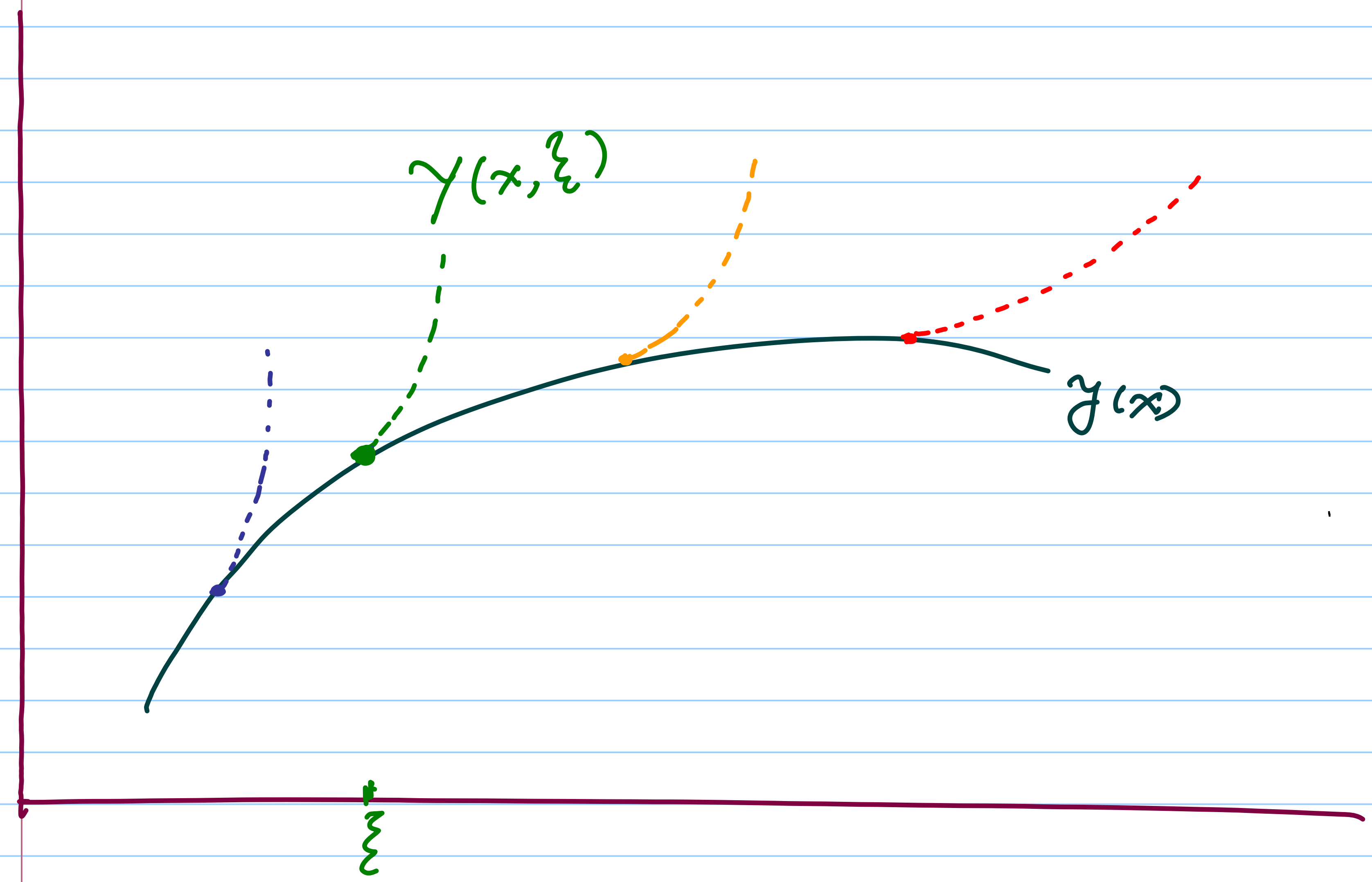
This must hold for all ξ (!)

Differentiate the first equation w.r.t ξ and subtract from the second and we get

$$c_1'(\xi)\, y_1(\xi) + c_2'(\xi)\, y_2(\xi) = 0 \quad (4.18)$$

We have precisely arrived at the Lagrangian Ansatz (!)

The seemingly ad hoc assumption (4.18) now aquires a significant geometrical interpretation. Namely that the true trajectory $y(x)$ is the "Envelope" of the family $\{Y(x;\xi)\}_{\xi}$ of fictitious trajectories and (4.18) is simply the tangency condition.



γ(x, ξ)
y(x)
ξ

End of Chapter $\overline{\underline{IV}}$.

Reference

www.math-cs.ucmo.edu/~mjms/

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Chapter V

Improper Integrals / Gamma Function

In what follows f is a real valued function that has only finitely many jump discontinuities on any <u>closed</u> <u>**bounded**</u> interval in $[a, \infty)$

We wish to assign a meaning to

$$\int_a^\infty f(x)\,dx. \tag{5-1}$$

(5-1) is called an improper integral of the **first kind.**

Note that we are assuming that $f(x)$ is bounded on $[a, T]$ for each $T > a$

and $\int_a^T f(x)\,dx$ exists as an ordinary Riemann-integral

(There are only finitely many points of discontinuities in $[a, T]$)

We define $\int_a^\infty f(x)\,dx$ to be

$$\lim_{T\to\infty} \int_a^T f(x)\,dx$$

(if the limit exists)

<u>Example:</u> $\int_0^\infty e^{-x}\,dx = 1$

Since $\int_0^T e^{-x}\,dx = 1 - e^{-T}$

<u>Exercise</u>: Show that if $n \in \mathbb{N}$

$$\int_0^\infty x^{n-1} e^{-x}\, dx = (n-1)! \qquad (5.2)$$

(Use induction. We have just finished the case $n=1$)

<u>Ex</u>:
$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{T\to\infty} \int_0^T \frac{dx}{1+x^2}$$

$$= \lim_{T\to\infty} (\tan^{-1} T - \tan^{-1} 0)$$

$$\therefore = \pi/2$$

In general it will not be possible to evaluate $\int_a^T f(x)\,dx$ explicitly

We need efficient methods to decide whether $\int_a^\infty f(x)\,dx$ exists.

Terminology: The improper integral of the first kind

$\int_a^\infty f(x)\,dx$ is said to <u>be Convergent</u>

if $\lim_{T\to\infty} \int_a^T f(x)\,dx$ exists.

Otherwise it is said to be <u>divergent</u>.

Definition: The improper integral

$\int_a^\infty f(x)\,dx$ is said to be

absolutely Convergent if

$\int_a^\infty |f(x)|\,dx$ converges.

Note: Absolute Convergence $\Rightarrow$ Convergence

Def: If $\int_a^\infty f(x)\,dx$ converges but $\int_a^\infty |f(x)|\,dx$ diverges then we say $\int_a^\infty f(x)\,dx$ is Conditionally Convergent.

We now state without proof the following

Theorem: (Comparison Test)

Suppose $|f(x)| \leq g(x)$ for all $x \geq a$ ($g(x)$ dominates $|f(x)|$)

and $\int_a^\infty g(x)\,dx$ Converges

Then $\int_a^\infty f(x)\,dx$ Converges absolutely

Example: $\int_0^\infty \frac{\sin x}{1+x^2}\,dx$ is absolutely

Convergent.

Note that $\displaystyle\int_1^{\infty} \frac{\cos x}{x^2}\,dx$ is

absolutely Convergent.

Let us prove that $\displaystyle\int_1^{\infty} \frac{\sin x}{x}\,dx$ is

Convergent.

First Consider $\displaystyle\int_1^{T} \frac{\sin x}{x}\,dx$. Integrate by parts:

$$\int_1^T \frac{\sin x}{x}\,dx = \int_1^T \frac{1}{x}\,\frac{d}{dx}(-\cos x)\,dx$$

$$= \cos 1 - \frac{1}{T}\cos T - \int_1^T \frac{\cos x}{x^2}\,dx$$

Now $\lim_{T\to\infty} \int_1^T \frac{\cos x}{x^2}\,dx$ exists since

we have seen that $\int_1^\infty \frac{\cos x}{x^2}\,dx$ is

absolutely convergent. Hence

$$\lim_{T\to\infty} \int_1^T \frac{\sin x}{x}\,dx = \lim_{T\to\infty}\left(\cos 1 - \frac{\cos T}{T}\right) + \lim_{T\to\infty}\int_1^T \frac{\cos x}{x^2}\,dx$$

exists

So $\displaystyle\int_1^\infty \frac{\sin x}{x}\,dx$ Converges

Fact: $\displaystyle\int_1^\infty \frac{|\sin x|}{x}\,dx$ diverges.

Thus $\displaystyle\int_1^\infty \frac{\sin x}{x}\,dx$ is Conditionally Convergent.

Note that $\frac{\sin x}{x}$ is continuous on $[0,1]$ if assigned the value 1 at origin.

So $\int_0^1 \frac{\sin x}{x}\,dx$ is a Proper Riemann integral. Together with the fact that $\int_1^\infty \frac{\sin x}{x}\,dx$ is convergent, we conclude

$$\int_0^\infty \frac{\sin x}{x}\,dx \text{ is convergent.}$$

(Conditionally convergent in fact)

We shall see later that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}. \qquad (5.3)$$

<u>Ex:</u> A direct application of Comparison test shows that

$$\int_0^\infty \frac{(\cos x)\, dx}{1+x^2}$$

is **Absolutely** Convergent.

We shall see that the value of this equals

$$\frac{\pi}{2e}$$

The ubiquitous Probability Integral

This is one of the most important of all integrals.

<u>Theorem</u>:

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \qquad (5.4)$$

<u>proof</u>: $\int_0^1 e^{-x^2}dx$ is a proper Riemann Int.

To prove that $\int_1^\infty e^{-x^2}dx$ is Convergent

note that $0 \leq e^{-x^2} < e^{-x}$; $x \geq 1$

and $\int_1^\infty e^{-x}\,dx$ converges. Hence

$\int_1^\infty e^{-x^2}\,dx$ converges by

Comparison test.

To determine the value of the integral,

$$I = \int_0^\infty e^{-x^2}\,dx$$

$$I = \int_0^\infty e^{-y^2}\,dy$$

$$\therefore \ I^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$$

put $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

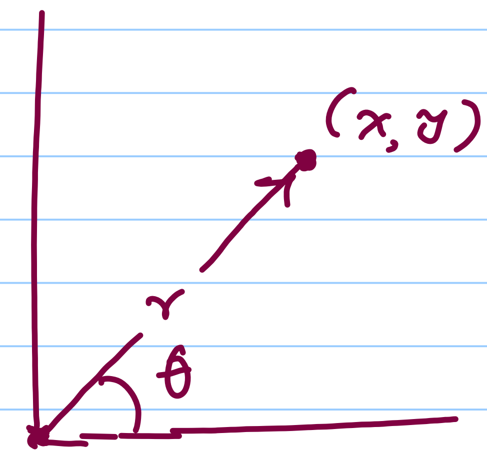
$$\therefore \ I^2 = \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} r\,dr\,d\theta$$

$$= \int_{r=0}^{\infty} r e^{-r^2} dr \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{4} \int_0^{\infty} 2r e^{-r^2} dr$$

$$= \frac{\pi}{4} \int_0^{\infty} e^{-t} dt = \frac{\pi}{4}$$

$$\therefore \ I = \frac{\sqrt{\pi}}{2}$$



Improper Integrals of the Second kind:

Consider the integral $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$

The integrand is Continuous on $[\epsilon, 1]$ for every $\epsilon > 0$. We define

$$\int_0^1 \frac{dx}{\sqrt{x}} \quad \text{as} \quad \lim_{\epsilon \to 0+} \int_\epsilon^1 \frac{dx}{\sqrt{x}}$$

$$= \lim_{\epsilon \to 0} 2(1 - \sqrt{\epsilon}) = 2.$$

More generally suppose that $f : (a, b] \to \mathbb{R}$ given $f(x)$ is Riemann integrable on $[T, b]$ for every T s.t. $a < T < b$ and

$$\lim_{T \to a} \int_T^b f(x)\,dx \text{ exists}$$

then we define

$$\int_a^b f(x)\,dx = \lim_{T \to a} \int_T^b f(x)\,dx$$

The improper integral $\int_a^b f(x)\,dx$ is said to be convergent

Notions like absolutely convergent
Conditionally convergent
may be defined exactly as was done for improper integrals of the first kind

For example

Theorem: (Comparison Test)

Suppose $|f(x)| \leq g(x)$ on $a < x \leq b$

and $\int_a^b g(x)\,dx$ is convergent then

$\int_a^b f(x)\,dx$ is absolutely convergent

As before

Absolute Convergence $\Rightarrow$ Convergence

but converse is not true.

<u>Example:</u>

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$ is an improper

integral of the second kind. Show that its

value is $\pi/2$

(III) Combination of both types:

Consider $$\int_0^\infty \frac{e^{-x}dx}{\sqrt{x}}$$

Break up the integral as

$$\int_0^1 \frac{e^{-x}dx}{\sqrt{x}} + \int_1^\infty \frac{e^{-x}dx}{\sqrt{x}}$$

and analyze the two pieces separately:

$\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{x}}dx$ is convergent because

$0 \le \dfrac{e^{-x}}{\sqrt{x}} < \dfrac{1}{\sqrt{x}}$ and $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ Converges

By Comparison test $\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$ Converges

For the second piece $\displaystyle\int_1^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$.

Note that $\dfrac{e^{-x}}{\sqrt{x}} < e^{-x}$ and $\displaystyle\int_1^\infty e^{-x}\,dx$

Converges. Again By Comparison test we Conclude that

$$\int_1^\infty \frac{e^{-x}}{\sqrt{x}}\,dx \text{ Converges.}$$

$\therefore \displaystyle\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$ is a convergent integral.

The Gamma Function: The example

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx = \int_0^\infty x^{\frac{1}{2}-1} e^{-x}\,dx$$

is a special case of the **gamma function** that we now describe

Exercise: Imitate the analysis employed for $\displaystyle\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$ to prove that

<u>If</u> $\underline{a > 0}$ then $\displaystyle\int_0^\infty x^{a-1} e^{-x}\,dx$ is

<u>Convergent</u>. Also show that when $a = 0$ the integral diverges.

<u>Note</u>: To deal with $\displaystyle\int_1^\infty x^{a-1} e^{-x}\,dx$

pick a $N > a$; $N \in \mathbb{N}$ and compare with $x^{N-1} e^{-x}$. You may recall an earlier exercise in which you were asked to show

$$\int_0^\infty x^{n-1} e^{-x}\,dx = (n-1)! \qquad (n \geq 1)$$

Definition: The Gamma Function $\Gamma(a)$ is defined for all $a > 0$ as

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} \, dx \qquad (5.5)$$

$$\Gamma(1) = 1$$

$$\Gamma(n) = (n-1)! \qquad (5.6)$$

<u>Exercise:</u> Check by integration by parts that

$$\Gamma(a+1) = a\,\Gamma(a) \qquad (5.7)$$

Let us calculate $\Gamma(\tfrac{1}{2})$.

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty x^{\frac{1}{2}-1} e^{-x}\,dx; \quad \text{put } x = t^2$$

$$= 2\int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \tag{5.8}$$

Thus

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{2}\,\Gamma\left(\tfrac{3}{2}\right) = \tfrac{3}{4}\sqrt{\pi} \quad \text{etc;}$$

These are the <u>ONLY</u> computable cases of $\Gamma(a)$

The gamma function is one of the most important <u>Special functions</u> of Mathematics.

It appears in numerous formulas in Mathematical Physics

Since $\Gamma(n) = (n-1)!$, the gamma function interpolates the factorial.

<u>The Beta Function</u>: This is a function of two variables defined as

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx \; ; \quad \begin{matrix} p>0 \\ q>0 \end{matrix}$$

Let us break the integral into two pieces

$$\int_0^{\frac{1}{2}} x^{p-1}(1-x)^{q-1}\,dx \quad \text{and}$$

$$\int_{\frac{1}{2}}^{1} x^{p-1}(1-x)^{q-1}\,dx$$

We must show that both are Convergent.

Take the first piece

$$\int_0^{1/2} x^{p-1}(1-x)^{q-1}\,dx$$

$$\frac{1}{2} \leq (1-x) \leq 1$$

So $\left(\frac{1}{2}\right)^{q-1} \leq (1-x)^{q-1} \leq 1$ if $q \geq 1$

Or $1 \leq (1-x)^{q-1} \leq \left(\frac{1}{2}\right)^{q-1}$ if $0 < q \leq 1$

In any case $(1-x)^{q-1} \leq M$

where $M = 1 + \left(\frac{1}{2}\right)^{q-1}$

and so

$$x^{p-1}(1-x)^{q-1} \leq M x^{p-1}$$

Now $\int_0^{1/2} M x^{p-1}\,dx$ Converges if $p > 0$

By Comparison test,

$$\int_0^{1/2} x^{p-1}(1-x)^{q-1}\,dx \quad \text{Converges.}$$

Likewise $\displaystyle\int_{\frac{1}{2}}^{1} x^{p-1}(1-x)^{q-1}\,dx$ Converges

$$\int_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

<u>Exercises</u>: $B(p,q) = B(q,p)$

$$B(p+1,q) = \frac{p}{q}\,B(p,q+1)$$

$$B(p+1,q) + B(p,q+1) = B(p,q)$$

$$B(p+1,q) = \frac{p}{p+q}\,B(p,q)$$

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$$

Theorem (Beta-Gamma Relation)

$$\Gamma(p)\,\Gamma(q) = \Gamma(p+q)\, B(p,q) \; ; \; p, q > 0$$

We shall prove this later.

Theorem: (Reflection Formula)

$$\Gamma(p)\,\Gamma(1-p) = \pi \operatorname{cosec} p\pi$$

$0 < p < 1$.

All known proofs of this formula are either tricky or use more advanced material. An elegant proof was given by Richard Dedekind in 1852 in his PhD dissertation (written under the supervision of Gauss)

We shall put up Dedekind's proof on the website.

Dedekind's proof is relevant in MA108 inasmuch as he obtains a second order ODE for the function $\Gamma(x)\Gamma(1-x)$ and integrates this to get the value $\frac{\pi}{\sin \pi x}$ (!)

Theorem:

$$\int_0^\infty \frac{x^{a-1}dx}{1+x} = \frac{\pi}{\sin \pi a}$$

$$0 < a < 1.$$

proof: put $t = \frac{1}{1+x}$. The integral becomes

$$\int_0^1 (1-t)^{a-1} \cdot t^{-a} \, dt$$

$$= B(a, 1-a)$$

$$= \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a}$$

Example: $\displaystyle\int_0^\infty \frac{dx}{1+x^4}$; $x^4 = u$

$$= \int_0^\infty \frac{1}{4} \frac{u^{1/4-1}}{1+u}\, du$$

$$= \frac{\pi}{4} \operatorname{cosec} \frac{\pi}{4} = \frac{\pi}{2\sqrt{2}}$$

Calculate $\displaystyle\int_0^\infty \frac{dx}{1+x^6}$; $\displaystyle\int_0^\infty \frac{dx}{1+x^{10}}$.

Numerous integrals reduce to Beta and Gamma integrals

Fresnel's Integrals:

These are the integrals

$$\int_0^\infty \sin x^2 \, dx \quad \text{and} \quad \int_0^\infty \cos x^2 \, dx$$

Generally let $a > 1$. We show that

$$\int_0^\infty \sin t^a \, dt \quad \text{and} \quad \int_0^\infty \cos t^a \, dt$$

Converge.

First of all $\int_0^1 \sin t^a \, dt$ is a PROPER

Riemann integral. It suffices to look at

$$\int_1^\infty \sin t^a \, dt. \quad \text{Put } t^a = u$$

$$\frac{1}{a} \int_1^\infty \frac{\sin u}{u^{1-1/a}} \, du \qquad \left(\begin{array}{c} a > 1 \text{ so} \\ 1 - \frac{1}{a} > 0 \end{array} \right)$$

Integrate by parts (Recall the analysis

of $\displaystyle\int_1^\infty \frac{\sin u}{u} \, du$)

Exercises: Discuss (i) $\int_0^1 \ln x \, dx$

(ii) $\int_0^\infty \frac{1}{x^2}\left(e^{-ax^2} - e^{-bx^2}\right) dx$

$0 < a < b$

Evaluate the integral

(iii) Discuss for convergence

$\int_1^\infty \ln\left(1 + \frac{1}{x^2}\right) dx$ and $\int_0^1 \ln\left(1 + \frac{1}{x^2}\right) dx$

(iv) Show that $\int_0^\infty \frac{x^2\,dx}{1+x^4}$ and $\int_0^\infty \frac{dx}{1+x^4}$ both converge. Find their values.

(v) Show that $\int_0^\infty \frac{dx}{\sqrt{1+x^3}}$ and $\int_1^\infty \frac{dx}{\sqrt{x^3-1}}$ converge. Reduce them to Beta integrals

(vi) Show that $\int_1^\infty \frac{dx}{(x^4-x)^{1/3}}$ converges. Reduce it to a beta-integral.

(vii) Show that $\displaystyle\int_1^\infty \frac{dx}{x\,(x^4-x^3)^{1/4}}$ converges. Find its value.

End of Chapter $\overline{\underline{V}}$

Chapter VI Laplace Transforms

This is the most important chapter in the course.

Def: A function $f : [0, \infty) \to \mathbb{R}$ is said to be of exponential type if

(i) f has only finitely many jump discontinuities in $[0, T]$ for each $T > 0$

(ii) There exists constants M, ω, T_0 such that

$$|f(t)| \leq M e^{\omega t} \quad \text{for all } t \geq T_0$$

That is to say f is dominated by an exponential function

<u>Note:</u> Sometimes we may have to work with functions that become infinite like $\frac{1}{\sqrt{x}}$. We demand in place of (i) that

$$\int_0^T |f(t)| \text{ converges for each } T.$$

We shall assume that there are ONLY FINITELY POINTS at which $f(t)$ becomes infinite.

For example $\dfrac{1}{\sqrt{t|1-t|}}$ is an admissible function. It is of exponential type and $\int_0^T |f(t)|$ converges for each $T > 0$.

<u>Def:</u> If $f(t)$ is of exponential type, the Laplace transform of $f(t)$ denoted by $\mathcal{L}f(s)$ or $F(s)$ is defined to be

$$\mathcal{L}f(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt \qquad (6.1)$$

We shall use uppercase letters for the Laplace transform. Thus $F(s)$, $G(s)$, $X(s)$, $Y(s)$ etc; would denote the Laplace transforms of $f(t)$, $g(t)$, $x(t)$ or $y(t)$.

<u>Examples:</u> $$\mathcal{L}\,1 = \int_0^\infty e^{-st}\,dt = \frac{1}{s}$$

$$\mathcal{L}\,t = \int_0^\infty t\,e^{-st}\,dt = \frac{1}{s^2}$$

Generally $\quad \mathcal{L}\, t^n = \int_0^\infty t^n e^{-st}\, dt = \frac{n!}{s^{n+1}}$. $\quad$ (6.2)

Theorem:

If a is a real number > -1

$$\mathcal{L}\, t^a = \frac{\Gamma(a+1)}{s^{a+1}} \quad \text{(Check)} \qquad \text{——} (6.3)$$

Thus $\quad \mathcal{L}\left(\frac{1}{\sqrt{t}}\right) = \frac{\sqrt{\pi}}{\sqrt{s}}$

Examples with Trigonometric Functions

$$\left.\begin{aligned} \mathcal{L}(\cos \omega t) &= \frac{s}{s^2+\omega^2} \\ \mathcal{L}(\sin \omega t) &= \frac{\omega}{s^2+\omega^2} \end{aligned}\right\} \qquad (6.4)$$

Check these by computing the integrals

$$\int_0^\infty (\sin \omega t)\, e^{-st}\, dt \quad \text{and} \quad \int_0^\infty (\cos \omega t)\, e^{-st}\, dt$$

Recall that if $a \in \mathbb{C}$

$$\begin{aligned} t^a &= \exp(a \ln t) \qquad (t > 0) \\ &= t^s(\cos \omega \ln t + i \sin \omega \ln t) \end{aligned}$$

$$\mathcal{L}\, t^a = \int_0^\infty t^a e^{-st}\, dt$$

$$= \int_0^\infty t^\beta (\cos \omega \ln t)\, e^{-st}\, dt + i \int_0^\infty t^\beta (\sin \omega \ln t)\, e^{-st}\, dt$$

Exponential Functions

$$\mathcal{L}\, e^{at} = \int_0^\infty e^{at-st}\, dt = \frac{1}{s-a} \quad (s > a) \qquad (6.5)$$

Evaluate $\mathcal{L}(t e^{at})$:

$$\mathcal{L}(te^{at}) = \int_0^\infty t e^{(a-s)t} \, dt$$

put $(s-a)t = u$ $\quad (s > a)$

$$= \int_0^\infty u e^{-u} \frac{du}{(s-a)^2}$$

$$= \frac{\Gamma(2)}{(s-a)^2} = \frac{1}{(s-a)^2}$$

<u>Note:</u> The improper integral converges <u>Only</u> for $s > a$. The Laplace transform

makes sense only for $s > a$

Generally if f is of exponential type $F(s)$, the Laplace transform of f is defined on a certain ray (ω, ∞)

It is TEMPTING to extend the formula

$$\mathcal{L} e^{at} = \frac{1}{s-a}$$

to complex values of a and then Deduce from it

$$\mathcal{L}(\sin at) = \mathcal{L}\left(\frac{1}{2i}\left(e^{ait} - e^{-ita}\right)\right)$$

$$= \frac{1}{2i}\left\{\frac{1}{s-ia} - \frac{1}{s+ia}\right\}$$

$$= \frac{a}{s^2+a^2}$$

The formula

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$ is <u>CORRECT</u>.

for complex a as well but our derivation for real a <u>does</u> <u>not</u> go through (!)

<u>Reason</u>: We made a substitution

$$(s-a)t = u$$

This would now be a Complex Substitution in a <u>real</u> <u>integral</u>. Such Substitutions Can lead to wrong results. Such as the following:

$$I = \int_0^\infty \frac{dx}{1+x^4}$$

"Put $x = iy$" and we get

$$I = iI \quad \text{or} \quad I = 0$$ which is plainly False!!

However the Formula

$\mathcal{L}\, e^{at} = \dfrac{1}{s-a}$ is valid

for complex a and you are allowed to use it for complex a

Differentiating the equation

$$\mathcal{L}\, e^{at} = \frac{1}{s-a}$$

w.r.t a gives

$$\mathcal{L}\, t\, e^{at} = \frac{1}{(s-a)^2}$$

$$\mathcal{L}(t^2 e^{at}) = \frac{2}{(s-a)^3} \quad \text{etc};$$

Thus
$$\mathcal{L}(t^k e^{at}) = \frac{k!}{(s-a)^{k+1}} \tag{6-6}$$

Relation (6.6) holds even when k is not an integer with the modification

$$\mathcal{L}(t^k e^{at}) = \frac{\Gamma(k+1)}{(s-a)^{k+1}} \tag{6.7}$$

We have the following table of Laplace Transforms summarizing the results obtained so far.

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
$\ldots$	$\ldots$
t^a	$\Gamma(a+1)/s^{a+1}$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$s/(s^2+\omega^2)$
$t \cos \omega t$	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$ (check)
$t \sin \omega t$	$2\omega s/(s^2+\omega^2)^2$
$\ldots$	$\ldots$ (??)

$f(t)$	$F(s)$	
e^{at}	$1/(s-a)$	
$t\,e^{at}$	$1/(s-a)^2$	
$t^2 e^{at}$	$2/(s-a)^3$	
$\dots \quad \dots$	$\dots \quad \dots$	
$t^k e^{at}$	$\dfrac{\Gamma(k+1)}{(s-a)^{k+1}}$	$(k > -1)$ $a \in \mathbb{C}$
$t^k e^{at} \cos \omega t$	??	

Decay of $\mathcal{L}f$ and local smoothness of $f(t)$

Looking at the table, we see

$\mathcal{L}1$ and $\mathcal{L}\cos t$ behave like $\frac{1}{s}$ for $s \gg 1$

$\mathcal{L}t$ and $\mathcal{L}\sin t$ behave like $\frac{1}{s^2}$ for $s \gg 1$.

$\mathcal{L}\sqrt{t}$ behaves like $\frac{1}{s^{3/2}}$

Whereas $\mathcal{L}\left(\frac{1}{\sqrt{t}}\right)$ behaves like $\frac{1}{\sqrt{s}}$.

We see that $\mathcal{L}f(s) \to 0$ as $s \to \infty$
but the rate of decay is different.

<u>Que:</u> Why is it that $\mathcal{L}\sin t$ decays faster than

$\mathcal{L}\cos t$?

Note that in the computation of $\mathcal{L}f$ we are "cutting off" the function (signal) for negative values of t

This causes a jump discontinuity to the graph of $\cos t$ but not of $\sin t$

However $f(t) = \begin{cases} \sin t & t \geq 0 \\ 0 & t \leq 0 \end{cases}$

has one sided derivatives at the origin that are unequal.

$$f(t) = \begin{cases} t^2 \text{ or } t\sin t & t \geq 0 \\ 0 & t \leq 0 \end{cases}$$

is differentiable at 0 but not twice diff.

$\mathcal{L}\, t^2$ and $\mathcal{L}\, t\sin t$ decay like $1/s^3$

$$f(t) = \begin{cases} \sqrt{t} & t \geq 0 \\ 0 & t \leq 0 \end{cases}$$ is merely continuous at 0 but doesn't have one sided derivatives.

$\mathcal{L}\sqrt{t}$ decays faster than $\mathcal{L}\,1$ and $\mathcal{L}\cos t$ but slower than $\mathcal{L}\sin t$

The function $$f(t) = \begin{cases} 1/\sqrt{t} & t > 0 \\ 0 & t < 0 \end{cases}$$

Not only is discontinuous but not even bounded in a nbd of the origin

This explains the poor decay $1/\sqrt{s}$

Which is worse than the $1/s$ decay of $\mathcal{L}\,1$ or $\mathcal{L}\cos t$.

Observation: The smoother the function, the faster is the decay (as $s \to \infty$) of the Laplace transform.

Similar results holds for Fourier transforms and Fourier series as well.

<u>Remark</u>: The preceeding observations are qualitative. It is quite non-trivial to describe the precise connection between the order of differentiability of $f(t)$ and the

decay property (as $s \to \infty$) of $\mathcal{L}f(s)$

The most basic result in this direction is the following

<u>Riemann - Lebesgue Lemma:</u>

Assume that

(i) For each $T > 0$, $\int_0^T |f(t)|\,dt$ exists either as a proper or improper integral.

(ii) $f(t)$ is of exponential type:

$$|f(t)| \le M e^{\omega t} \; ; \quad t \geq a$$

for some constants M, ω, a.

Then $\lim\limits_{s \to \infty} \mathcal{L}f(s) = 0$

<u>Proof</u>: $(\mathcal{L}f)(s) = \int_0^a f(t) e^{-st} dt$

$$+ \int_a^\infty f(t) e^{-st} dt$$

$\therefore |\mathcal{L}f(s)| \leq \int_0^a |f(t)| e^{-st} dt +$

$$\int_a^\infty |f(t)| e^{-st} dt$$

We show that both integrals on RHS have limit zero as $s \to \infty$

Consider the second piece $\int_a^\infty |f(t)| e^{-st}\, dt$

$$|f(t)|\, e^{-st} \leq M e^{\omega t} e^{-st} \; ; \quad t \geq a$$

$$\therefore \int_a^\infty |f(t)| e^{-st}\, dt \leq M/_{s-\omega} \longrightarrow 0 \text{ as } s \to \infty.$$

Now take the first piece $\int_0^a e^{-st} |f(t)|\, dt$

$$\int_0^a |f(t)| e^{-st} \, dt = \int_0^{1/\sqrt{s}} |f(t)| e^{-st} \, dt + \int_{1/\sqrt{s}}^a |f(t)| e^{-st} \, dt$$

$$\leq \int_0^{1/\sqrt{s}} |f(t)| \, dt + e^{-\sqrt{s}} \int_{1/\sqrt{s}}^a |f(t)| \, dt$$

$$\leq \left(\int_0^a |f(t)|\,dt - \int_{1/\sqrt{s}}^a |f(t)|\,dt \right)$$

$$+ \left(e^{-\sqrt{s}} \int_0^a |f(t)|\,dt \right) \qquad (*)$$

now $\displaystyle\lim_{\varepsilon \to 0} \int_\epsilon^a |f(t)|\,dt = \int_0^a |f(t)|\,dt$.

So $\displaystyle\lim_{s \to \infty} \int_{1/\sqrt{s}}^a |f(t)|\,dt = \int_0^a |f(t)|\,dt$

So the first term on RHS of $(*)$ goes to 0.
The second term trivially goes to zero

exponentially fast. This completes the proof of this result.

N.B.: There is an analogue of this for Fourier Series and Fourier transforms. Any of these results is popularly known as a Riemann-Lebesgue lemma.

Examples: Q15 on p10 of Tut Sheet.

$$f(t) = \int_0^\infty \frac{\sin tx}{x}\, dx \qquad (t > 0)$$

Laplace transforming we get-

$$F(s) = \int_0^\infty \frac{1}{x} \mathcal{L}(\sin tx)\, dx$$

$$= \int_0^\infty \frac{1}{x} \cdot \frac{x}{x^2+s^2}\, dx$$

$$= \int_0^\infty \frac{dx}{x^2+s^2} = \frac{1}{s} \tan^{-1}\left(\frac{x}{s}\right)\Big|_0^\infty$$

$$= \frac{\pi}{2s} = \mathcal{L}\left(\frac{\pi}{2}\right)$$

$$\therefore \quad f(t) = \frac{\pi}{2} \qquad (t>0)$$

(iii) $f(t) = \int_0^\infty \sin tx^a\, dx \; ; \qquad a > 1$

$$F(s) = \int_0^\infty \frac{x^a \, dx}{s^2 + x^{2a}}$$

put $x^{2a} = s^2 u$

$$F(s) = \int_0^\infty \frac{s\sqrt{u}}{s^2(1+u)} \cdot \frac{s^{1/a}}{2a} \frac{u^{\frac{1}{2a}-1}}{} du$$

$$= \frac{1}{s^{1-\frac{1}{a}}} \cdot \frac{1}{2a} \int_0^\infty \frac{u^{\frac{1}{2}+\frac{1}{2a}-1}}{1+u} du$$

$$= \frac{1}{2a} \cdot \frac{1}{s^{1-\frac{1}{a}}} \pi \operatorname{cosec}\left(\frac{\pi}{2} + \frac{\pi}{2a}\right)$$

$$= \frac{\pi}{2a} \cdot \frac{1}{s^{1-1/a}} \cdot \sec\left(\frac{\pi}{2a}\right)$$

Recall $\mathcal{L}\, t^{k} = \dfrac{\Gamma(k+1)}{s^{k+1}}$

we see that

$$f(t) = \frac{\pi}{2a} \sec\left(\frac{\pi}{2a}\right) \frac{t^{-1/a}}{\Gamma\left(1-\frac{1}{a}\right)}$$

In particular

$$\int_0^\infty \sin x^a \, dx = \frac{\pi}{2a} \sec\left(\frac{\pi}{2a}\right) \cdot \frac{1}{\Gamma\left(1-\frac{1}{a}\right)}$$

Fresnel Integral : $\displaystyle\int_0^\infty \sin x^2 \, dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$

Show that $\displaystyle\int_0^\infty \cos x^2 \, dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$

Do Q15 (v) on page 10 and then

Do Q15 (ii), (iv) and (vi) in <u>that</u> order.

Shift Theorem:

If $\mathcal{L}f = F(s)$ then

$$\mathcal{L}(e^{at}f(t)) = F(s-a)$$

Pf.: $\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-(s-a)t} f(t)\,dt$

$= F(s-a)$

Thus: $\mathcal{L}(e^{at}\cos\omega t) = \dfrac{s-a}{(s-a)^2+\omega^2}$

$\mathcal{L}(e^{at}\sin\omega t) = \dfrac{\omega}{(s-a)^2+\omega^2}$ etc.;

Some more examples:

Q2 (iv) on page 8

Find the inverse Laplace transform of

$$\frac{s^3}{s^4+4a^4}$$

$$s^4+4a^4 = (s^2-2as+2a^2)(s^2+2as+2a^2)$$

$$\frac{s^3}{s^4+4a^4} = \frac{As+B}{s^2-2as+2a^2} + \frac{Cs+D}{s^2+2as+2a^2}$$

$$s^3 = (As+B)(s^2+2as+2a^2) + (Cs+D)(s^2-2as+2a^2)$$

Comparing coeff of $s^3, s^2, s, 1$,

$$A + C = 1$$
$$B + D = 0$$
$$A - C = 0 \qquad \therefore \; A = C = \frac{1}{2}$$
$$B - D = -a \qquad B = -D = -a/2$$

$$\therefore \frac{s^3}{s^4 + 4a^4} = \frac{1}{2}\left(\frac{s-a}{s^2 - 2as + 2a^2}\right) + \frac{1}{2}\left(\frac{s+a}{s^2 + 2as + 2a^2}\right)$$

$$= \frac{1}{2}\left(\frac{s-a}{(s-a)^2 + a^2}\right) + \frac{1}{2}\left(\frac{s+a}{(s+a)^2 + a^2}\right)$$

$$= \frac{1}{2}\mathcal{L}\; e^{at}\cos at + \frac{1}{2}\mathcal{L}\left(e^{-at}\cos at\right)$$

$$= \mathcal{L}\left((\cos at)\left(\frac{e^{at} + e^{-at}}{2}\right)\right)$$

Thus $\mathcal{L}^{-1}\left(\dfrac{s^3}{s^4+4a^4}\right) = \dfrac{(\cos at)(e^{at}+e^{-at})}{2}$

<u>Aliter:</u> One could have used the Complex form for partial fraction decomposition.

For example, Compute $\mathcal{L}^{-1}\left(\dfrac{1}{1+s^4}\right)$

$$\frac{1}{1+s^4} = \frac{a_1}{s-\omega_1} + \frac{a_2}{s-\omega_2} + \frac{a_3}{s-\omega_3} + \frac{a_4}{s-\omega_4}.$$

where $\omega_1, \omega_2, \omega_3, \omega_4$ are the roots of

$$s^4+1=0$$

Multiply by $s-\omega_j$ and let $s \to \omega_j$:

$$a_j = \lim_{s \to \omega_j} \frac{s - \omega_j}{1 + s^4} = \frac{1}{4\omega_j^3}$$

$$a_j = \frac{\omega_j}{4\omega_j^4} = -\frac{\omega_j}{4} \quad (!)$$

Take $\omega_1 = e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$$\omega_2 = \overline{\omega_1} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$\omega_3 = \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \,; \quad \omega_4 = \frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}} = \overline{\omega_3}$$

$$\frac{1}{1+s^4} = \frac{\omega_1}{s-\omega_1} + \frac{\bar{\omega}_1}{s-\bar{\omega}_1} + \frac{\omega_3}{s-\omega_3} + \frac{\bar{\omega}_3}{s-\bar{\omega}_3}$$

$$= \mathcal{L}\left(\omega_1 e^{\omega_1 t} + \bar{\omega}_1 e^{\bar{\omega}_1 t}\right)$$

$$+ \mathcal{L}\left(\omega_3 e^{\omega_3 t} + \bar{\omega}_3 e^{\bar{\omega}_3 t}\right)$$

$$\mathcal{L}^{-1}\left(\frac{1}{1+s^4}\right) = \frac{1}{\sqrt{2}}\left(e^{\omega_1 t} + e^{\bar{\omega}_1 t}\right)$$

$$+ \frac{i}{\sqrt{2}}\left(e^{\omega_1 t} - e^{\bar{\omega}_1 t}\right)$$

$$-\frac{1}{\sqrt{2}}\left(e^{\omega_3 t} + e^{\bar{\omega}_3 t}\right) + \frac{i}{\sqrt{2}}\left(e^{\omega_3 t} - e^{\bar{\omega}_3 t}\right)$$

$$= \sqrt{2}\, e^{t/\sqrt{2}} \cdot \cos t/\sqrt{2} - \sqrt{2}\, e^{t/\sqrt{2}} \sin t/\sqrt{2}$$

$$- \sqrt{2}\, e^{-t/\sqrt{2}} \cos t/\sqrt{2} - \sqrt{2}\, e^{-t/\sqrt{2}} \sin t/\sqrt{2}.$$

Example: Find $\mathcal{L}^{-1} \dfrac{s^2}{(s^2+1)^2}$ and

$$\mathcal{L}^{-1} \left(\frac{1}{(s^2+1)^2} \right)$$

Find $\mathcal{L}^{-1} \dfrac{1}{(s^2+1)^3}$ etc;

Some Rules for partial fraction decomposition

$\dfrac{P(s)}{Q(s)}$ (given Rational function)

$\deg P(s) \lneq \deg Q(s)$

$\gcd(P(s), Q(s) = 1$

If ω is a simple root of $Q(s)$ then

$$\frac{P(s)}{Q(s)} = \frac{a}{s-\omega} + \ldots ; \qquad a = \frac{P(\omega)}{Q'(\omega)}$$

If ω is a double root,

$$\frac{P(s)}{Q(s)} = \frac{a}{(s-\omega)^2} + \frac{b}{(s-\omega)} + \ldots ; \quad a = \frac{2P(\omega)}{Q''(\omega)}$$

$$b = \frac{d}{ds}\left(\frac{P(s)}{Q(s)} (s-\omega) \right)\Bigg|_{s=\omega}$$

$$\therefore \quad b = \frac{2P'(\omega)}{Q''(\omega)} + P(\omega)\frac{d}{ds}\left(\frac{(s-\omega)^2}{Q(s)}\right)\Bigg|_{s=\omega}$$

(One can write the second term in closed form involving derivatives of Q but it is tricky)

The most important case is the case of simple roots.

Derivatives:

<u>Thm:</u> $\mathcal{L}(f'(t)) = sF(s) - f(0)$

$$\mathcal{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}(f'''(t)) = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

etc.

<u>proof:</u> $\mathcal{L}f'(t) = \int_0^\infty e^{-ts} f'(t)\,dt$

$$= f(t)e^{-st}\Big|_0^\infty - \int f(t)\frac{d}{dt}e^{-st}\,dt.$$

$$= -f(0) + s\int_0^\infty f(t)\,e^{-st}\,dt$$

$$= s\,F(s) - f(0)$$

$$\mathcal{L}(f'') = s\,\mathcal{L}(f') - f'(0)$$

$$= s(s\mathcal{L}f - f(0)) - f'(0)$$

$$= s^2 F(s) - sf(0) - f'(0)$$

Likewise

$$\mathcal{L}f''' = s\,\mathcal{L}f'' - f''(0)$$

$$= s(s\mathcal{L}f' - f'(0) - f''(0)$$

$$= s^2(s\mathcal{L}f - f(0)) - sf'(0) - f''(0)$$

$$= s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

etc;

Applications to Differential Equations

Solve $\left.\begin{aligned} y'' + y &= \sin x \\ y(0) = 1, \quad & y'(0) = 0 \end{aligned}\right\} \quad (6\text{-}8)$

Taking the Laplace transform on both sides

$$s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{1}{s^2+1}$$

$$\therefore (s^2+1)\,Y(s) = s + \frac{1}{s^2+1}$$

$$Y(s) = \frac{s}{s^2+1} + \frac{1}{(s^2+1)^2}$$

or $\quad y(s) = \cos t - \mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right)$

$\mathcal{L}^{-1}F$ is that function whose Laplace transform is $F(s)$ (Like the anti derivative of elementary Calculus)

Let us use partial fractions:

$$\frac{1}{(s^2+1)^2} = \frac{1}{(s-i)^2} \cdot \frac{1}{(s+i)^2}$$

$$= \frac{A}{(s-i)^2} + \frac{B}{s-i} + \frac{C}{(s+i)^2} + \frac{D}{s+i}$$

We see that $A = \left(\frac{1}{2i}\right)^2 = -\frac{1}{4}$

$$C = \frac{1}{(-2i)^2} = -\frac{1}{4}$$

Mult by $(s-i)^2$, differentiate and put $s = i$:

$$B = \frac{d}{ds} \frac{1}{(s+i)^2} \Big|_{s=i} = \frac{-2}{(2i)^3}$$

$$= \frac{1}{4i}$$

Check that $D = -\frac{1}{4i}$

$$\frac{1}{(s^2+1)^2} = -\frac{1}{4} \left\{ \frac{1}{(s-i)^2} + \frac{1}{(s+i)^2} \right\} + \frac{1}{4i} \left(\frac{1}{s-i} - \frac{1}{s+i} \right)$$

Now recall $\mathcal{L}(te^{at}) = \dfrac{1}{s-a}$

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = -\frac{t}{4}\left(e^{it}+e^{-it}\right) + \frac{1}{4i}\left(e^{it}-e^{-it}\right)$$

$$= -\frac{t}{2}\cos t + \frac{1}{2}\sin t$$

This may look tedious but the method of Laplace Transforms can handle systems of ODEs with equal ease as we now illustrate.

It is important to note that the method is superior to other methods inasmuch as the initial conditions are automatically incorporated in the solution procedure.

Solving systems of linear diff eq^{n}s:

Q4 (iv) in TutSheet page 9

$$\left.\begin{aligned} \frac{dx}{dt} &= 5x + 8y + 1 \\ \frac{dy}{dt} &= -6x - 9y + t \end{aligned}\right\} \quad (6.9)$$

$$x(0) = 4, \quad y(0) = -3$$

Taking the Laplace transform of both sides in (6.9) we get

$$sX(s) - x(0) = 5X + 8Y + \frac{1}{s}$$

$$sY(s) - y(0) = -6X - 9Y + \frac{1}{s^2}$$

$$\therefore (s-5)X(s) - 8Y(s) = 4 + \frac{1}{s}$$

$$6X + (9+s)Y(s) = -3 + \frac{1}{s^2}$$

Solve for X, Y and find the functions $x(t)$ and $y(t)$

Note that the method can be applied to systems of higher order equations as well

Ex: Q4(vi) on p9:

$$y_1'' + y_2 = -5\cos 2t \; ; \; y_2'' + y_1 = 5\cos 2t$$

$y_1(0) = 1, \quad y_1'(0) = 1, \quad y_2(0) = -1, \quad y_2'(0) = 1$

<u>Work this out.</u>

Some more examples:

Q18 and Q20 in Tutorial Sheet.

Do Q18 yourself.

<u>Q20:</u> $\mathcal{L}\log t = \int_0^\infty e^{-st} \ln t \, dt$

put $st = u$

$$\therefore \mathcal{L}\ln t = \int_0^\infty e^{-u} \ln\left(\frac{u}{s}\right) \frac{du}{s}$$

$$= \int_0^\infty e^{-u} (\ln u - \ln s) \frac{du}{s}$$

$$= \frac{-\gamma}{s} - \frac{\ln s}{s} \qquad (6.11)$$

where the constant γ is given by

$$\gamma = -\int_0^\infty e^{-u} \ln u \, du \qquad (6.12)$$

(Called Euler's Constant)

Recall

$$\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} \, dt$$

$$\therefore \quad \Gamma'(a) = \int_0^\infty e^{-t} \cdot t^{a-1} \cdot \ln t \, dt$$

put $a=1$.

$$\Gamma'(1) = \int_0^\infty e^{-t} \ln t \, dt$$

Thus $\quad \gamma = -\Gamma'(1) \qquad (6.13)$

There is another representation of γ:

$$\int_0^\infty e^{-jt} \, dt = \frac{1}{j}$$

$$\therefore \; 1 + \frac{1}{2} + \dots + \frac{1}{N} = \int_0^\infty \sum_{j=1}^{N} e^{-jt} \, dt$$

$$\ln N = \int_0^\infty \frac{1}{t} \left(e^{-t} - e^{-Nt} \right) dt$$

Take the difference and let $N \to \infty$

$$\lim_{N\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{N}-\ln N\right)$$

$$= \int_0^\infty \left(\frac{1}{1-e^{-t}}-\frac{1}{t}\right)e^{-t}\,dt \qquad (6\text{-}14)$$

<u>Exercise</u>: Prove that

$$\int_0^\infty \left(\frac{1}{1-e^{-t}}-\frac{1}{t}\right)e^{-t}\,dt = -\int_0^\infty e^{-t}\ln t\,dt \qquad (6\text{-}15)$$

Laplace Transforms of Periodic functions.

Let $f(t)$ be periodic with period p.

$$f(t+p) = f(t)$$

$$\mathcal{L}f(t) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^p e^{-st} f(t)\,dt + \int_p^{2p} e^{-st} f(t)\,dt + \cdots$$

$$+ \int_{(n-1)p}^{np} e^{-st} f(t)\,dt + \cdots$$

$$= \sum_{n=1}^{\infty} \int_{(n-1)p}^{np} e^{-st} f(t)\,dt$$

put $t - (n-1)p = u$

$$= \sum_{n=1}^{\infty} \int_{0}^{p} e^{-su-(n-1)ps} f(u+(n-1)p)\,du$$

$$= \sum_{n=1}^{\infty} e^{-(n-1)ps} \cdot \int_{0}^{p} e^{-su} f(u)\,du$$

($\because f$ is periodic)

$$= \int_{0}^{p} e^{-su} f(u)\,du \Big/ (1-e^{-ps}) \qquad (6\text{-}16)$$

Ex:

Compute $\mathcal{L}(|\sin t|)$

$$\mathcal{L}(|\sin t|) = \frac{1}{(1-e^{-\pi s})} \int_0^{\pi} e^{-su} \sin u \, du$$

$$= \frac{1}{1+s^2} \cdot \frac{1+e^{-\pi s}}{1-e^{-\pi s}}$$

$$= \frac{1}{1+s^2} \coth \frac{\pi s}{2} \qquad (6\cdot 17)$$

Do Q6 on page 9.

Application: Periodic functions can be written as series of sines and cosines (Fourier Series). Let us write

If $f(t)$ is a 2π-periodic function

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \qquad (6.18)$$

We proceed formally and find a_j and b_j (Rigorous development of Fourier Analysis requires considerable knowledge of Analysis)

Multiply (6.18) by $\cos mt$ and integrate:

Note that $\displaystyle\int_{-\pi}^{\pi} \cos mt \cos nt = 0$ if $m \neq n$

$$\int_{-\pi}^{\pi} (\cos nt) \sin mt = 0$$

We get

$$\int_{-\pi}^{\pi} f(t) \cos mt \, dt = a_m \int_{-\pi}^{\pi} \cos^2 mt \, dt$$

(all other terms drop out)

$$= 2 a_m \int_{0}^{\pi} \cos^2 mt \, dt$$

$$= a_m \int_{0}^{\pi} (1 + \cos 2mt) \, dt$$

$$= \pi a_m$$

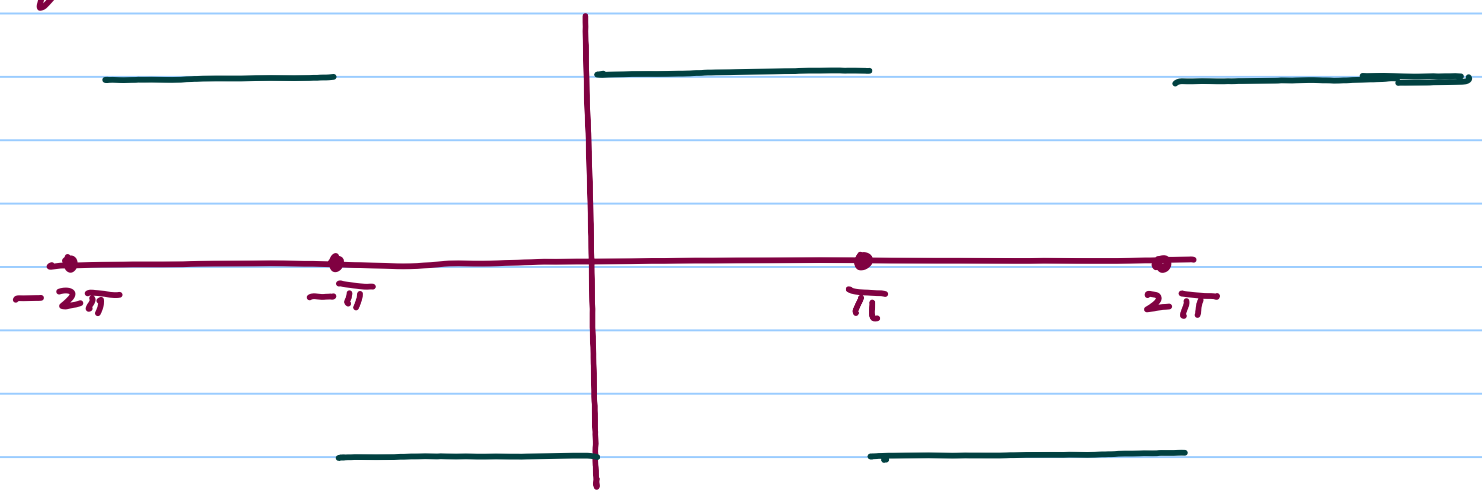
$$\therefore \quad a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos mt \, dt$$

Likewise

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin mt \, dt \qquad (6.19)$$

<u>Note:</u> $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$ obtained by integrating (6.18) directly.

Let us look at a simple example of a square wave train



$$f(t) = \begin{cases} 1 & \text{if } 0 < t < \pi \\ -1 & \text{if } -\pi < t < 0 \end{cases}$$

$f(t+2\pi) = f(t)$ $\quad 2\pi$- periodic

Since f is an odd function

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos mt\,dt = 0, \quad \text{also}$$

$$a_0 = 0$$

$$b_m = \frac{2}{\pi}\int_{0}^{\pi} f(t)\sin mt\,dt$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin mt\,dt = \frac{2}{\pi m}\left(1-(-1)^m\right)$$

Thus

$$f(t) = \sum_{m=1}^{\infty} \frac{2}{\pi m}\left(1-(-1)^m\right)\sin mt \qquad (6.20)$$

Note: The series representation (6.20) holds everywhere except at $0, \pm\pi, \pm 2\pi, \ldots$ (points of discontinuity of $f(t)$)

Let us now Laplace transform equation (6.20)

$$\frac{1}{1-e^{-2\pi s}} \cdot \int_0^{2\pi} f(t)\, e^{-st}\, dt = \sum_{m=1}^{\infty} \frac{2}{\pi m}\left(1-(-1)^m\right) \cdot \frac{m}{m^2+s^2}$$

$$= \frac{2}{\pi} \sum_{m=1}^{\infty} (1-(-1)^m) / (m^2+s^2)$$

Hence

$$\frac{1}{1-e^{-2\pi s}} \therefore \frac{1}{s} (1-e^{-\pi s})^2$$

$$= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{s^2+(2k-1)^2}$$

$$\therefore \frac{1}{s} \left(\frac{1-e^{-\pi s}}{1+e^{-\pi s}} \right) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{s^2+(2k-1)^2}$$

$$\frac{1}{s} \tanh\left(\frac{\pi s}{2}\right) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{s^2+(2k-1)^2}$$

$$\frac{\tanh \pi s}{s} = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{4s^2 + (2k-1)^2}$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right); \quad \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\sin i\theta = i\sinh\theta; \quad \cos i\theta = \cosh\theta$$

$$\tan i\theta = i\tanh\theta$$

$$\therefore \tan\theta = -i\tanh i\theta$$

$$\therefore \frac{\tan \pi s}{s} = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{-4s^2 + (2k-1)^2}$$

$$\tan \pi s = \frac{2s}{\pi} \sum_{k=1}^{\infty} \frac{1}{-s^2 + \left(\frac{2k-1}{2}\right)^2}$$

$$\tan s = \sum_{k=1}^{\infty} \frac{2s}{-s^2 + \left(\frac{2k-1}{2}\pi\right)^2} \qquad (6.21)$$

Mittag-Leffler Expansion for tan.

One can obtain similarly Mittag-Leffler expansions for other Trig functions

<u>Exercise:</u> $f(x) = x$ if $-\pi < x < \pi$

(1) Extend f as a 2π-periodic fun^n

$$f(x+2\pi) = f(x)$$

Determine the Fourier Series of $f(x)$.
Use Laplace transforms to deduce the Mittag-Leffler Expansion for

Coth πs and deduce the Mittag-Leffler expansion of Cot s

(2) Compute the Fourier Series of

$$f(x) = \begin{cases} \pi - x & 0 < x < \pi \\ x - \pi & -\pi < x < 0. \end{cases}$$

Use Laplace transforms to determine the Mittag Leffler expansion for $\operatorname{cosec} s$

(3) Let $f(x) = \sinh ax \quad ; \quad -\pi < x < \pi$

Extend $f(x)$ as a periodic function

$$f(x+2\pi) = f(x) \quad \text{for all } x.$$

Determine the Fourier Series for $f(x)$

This is valid at points $x \neq \pm\pi, \pm 3\pi, \ldots$

That is except at Points of discontinuity of $f(x)$.

put $x = \frac{\pi}{2}$ and deduce

$$\frac{\pi}{4} \sec a\pi = \sum \frac{(-1)^k \, (2k+1)^2}{(2k+1)^2 - 4a^2}.$$

Second Shift Theorem:

Heaviside unit Step Function $H(t)$: This is defined as

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$$

<u>Theorem:</u> If $a > 0$

$$\mathcal{L}(f(t-a)\,H(t-a)) = e^{-as}\,\mathcal{L}f$$

<u>pf:</u> $\mathcal{L}(f(t-a)\;H(t-a))$

$$= \int_0^\infty f(t-a)\,H(t-a)\,e^{-st}\,dt$$

$$= \int_a^\infty f(t-a)\, e^{-st}\, dt. \quad \text{Put } t-a=u$$

$$= \int_0^\infty f(u)\, e^{-s(a+u)}\, du$$

$$= e^{-as} \int_0^\infty f(u)\, e^{-su}\, du$$

$$= e^{-as} \mathcal{L}f.$$

Q8 <u>on page 9</u>: find $f(t)$ given that

$$\mathcal{L}f = \frac{1}{s^2}\left(e^{-s} - e^{-2s} - e^{-3s} + e^{-4s}\right)$$

Solution: $\mathcal{L}\left(f(t-a)\,H(t-a)\right) = e^{-as}F(s)$

So $\mathcal{L}^{-1}\left(e^{-as}F(s)\right) = f(t-a)\,H(t-a)$

So $\mathcal{L}^{-1}\left(e^{-s}\frac{1}{s^2}\right) = (t-1)\,H(t-1)$

$(\because \mathcal{L}t = 1/s^2)$

Likewise $\mathcal{L}^{-1}\left(e^{-2s}\frac{1}{s^2}\right) = (t-2)\,H(t-2)$

$\mathcal{L}^{-1}\left(e^{-3s}\frac{1}{s^2}\right) = (t-3)\,H(t-3)$

$\mathcal{L}^{-1}\left(e^{-4s}\frac{1}{s^2}\right) = (t-4)\,H(t-4)$

The desired function $f(t)$ is given by.

$$f(t) = (t-1)H(t-1) - (t-2)H(t-2) - (t-3)H(t-3) + (t-4)H(t-4)$$

$$\therefore \quad f(t) = \begin{cases} 0 & \text{if} \quad t \leq 1 \\ t-1 & 1 \leq t \leq 2 \\ (t-1)-(t-2) = 1 \; ; & 2 \leq t \leq 3 \\ (t-1)-(t-2)-(t-3) = 4-t \; ; & 3 \leq t \leq 4 \\ 0 & \text{if} \quad t \geq 4 \end{cases}$$

<u>Remark:</u> Q9 on p 9

$u_\pi(t)$ is the Heaviside function $H(t-\pi)$ and $u_1(t)$ is the Heaviside function $H(t-1)$

Q10 on p9: Calculate $\mathcal{L}^{-1} \ln\left(\frac{s^2+4s+5}{s^2+2s+5}\right)$

$$f(t) = \mathcal{L}^{-1}\left(\ln\left(\frac{s^2+4s+5}{s^2+2s+5}\right)\right)$$

$$\mathcal{L} f(t) = \ln\left(\frac{s^2+4s+5}{s^2+2s+5}\right) = F(s)$$

$$\therefore \mathcal{L}(t f(t)) = -F'(s)$$

$$= \frac{2s+2}{s^2+2s+5} - \frac{2s+4}{s^2+4s+5}$$

$$t f(t) = 2\mathcal{L}^{-1}\left(\frac{s+1}{s^2+2s+5}\right) - 2\mathcal{L}^{-1}\left(\frac{s+2}{s^2+4s+5}\right)$$

$$= 2\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+4}\right) - 2\mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+1}\right)$$

Complete the Solution.

Q12 on page 10: $Y(s)$ is the Laplace transform of a solution of

$$ty'' + y' + ty = 0 \quad ; \quad t > 0 \quad ; \quad y(0) = k$$

(Bessel's function of order zero) ; $Y(1) = \frac{1}{\sqrt{2}}$

$$\mathcal{L}(ty'') + \mathcal{L}(y') + \mathcal{L}(ty) = 0$$

$$-\frac{d}{ds}\mathcal{L}(y'') + sY - y(0) - \frac{d}{ds}Y = 0$$

$$\therefore -\frac{d}{ds}\left(s^2 Y - s y(0) - y'(0)\right) + sY - y(0) - \frac{d}{ds}Y(s) = 0$$

$\therefore -s^2Y' - sY - Y' = 0$

$\therefore (s^2+1)Y' + sY = 0$

Solve this linear ODE. Complete the Sol^n.

<u>Remark:</u> Q5 on page 9 stands Canceled.

Convolution of two functions:

Given functions $f(t)$ and $g(t)$ on the real line their Convolution $f * g$ is defined to be

$$(f * g)(t) = \int_{-\infty}^{\infty} f(x)\, g(t-x)\, dx \qquad (6.23)$$

Check that $f * g = g * f$ (in the sense that if the improper integral defining $f * g$ exists then so does the integral for $g * f$ and the two are equal)

Meaning of Convolution:

The $f(x)\,dx$ in the integrand should be viewed as a "density" against which the function g is being "averaged". Being an "average" the convolution will be better behaved than g (and also better than f due to symmetry).

Consider $f(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t \notin [0,1] \end{cases}$

$f(t)$ is discontinuous at 0 and 1.

Let us compute $f * f$

$$(f*f)(t) = \int_{-\infty}^{\infty} f(x) f(t-x)\, dx$$

$$= \int_{0}^{1} f(x) f(t-x)\, dx$$

since $f(x)$ vanishes outside $[0,1]$.

$$= \int_0^1 f(t-x)\,dx \qquad (6.24)$$

Now if $t < 0$ then $t - x < 0$ as x runs from 0 to 1 and the integral (6.24) vanishes

<u>If</u> $t > 2$ then $t - x > 1$ as x runs from 0 to 1 and again the integral (6.24) vanishes.

Determine the value of $f * f(t)$ when $0 \leq t \leq 2$.

Let us compute

$$t^a H(t) * t^b H(t):$$

$$= \int_{-\infty}^{\infty} x^a H(x) (t-x)^b H(t-x)\, dx$$

$$= \int_0^{\infty} x^a (t-x)^b H(t-x)\, dx$$

Now if $t<0$ then $H(t-x) \equiv 0$ for all $x>0$

So the integral vanishes

Thus $t^a H(t) * t^b H(t)$

$$= H(t) \int_0^\infty x^a (t-x)^b H(t-x)\, dx$$

$H(t-x) = 0$ when $x > t$

$$= H(t) \int_0^t x^a (t-x)^b\, dx$$

put $x = tu$

$$= H(t)\, t^{a+b} \int_0^1 u^a (1-u)^b\, t\, du$$

$$= B(a+1, b+1)\, t^{a+b+1} H(t)$$

Thus $\quad t^{a-1}H(t) * t^{b-1}H(t)$

$$= B(a,b)\, t^{a+b-1} H(t)$$

Convolution Theorem:

If $f(t)$ and $g(t)$ are functions of exponential type then

$$\mathcal{L}\big(H(t)f(t) * H(t)g(t)\big) = (\mathcal{L}f)(\mathcal{L}g)$$

<u>Proof</u>: Note first that

$$H(t)f(t) * H(t)g(t) = \int_0^\infty f(x)\, H(t-x)\, g(t-x)\, dx$$

$$= \int_0^t f(x)\, g(t-x)\, dx$$

We shall give a sketch to show that the convolution is of exponential type. The details can be fixed by the student.

One breaks the integral $\int_0^t$ into two pieces

$$\int_0^a + \int_a^t$$

In the first piece x is constrained to ~~stay~~ in $[0, a]$ and ~~so~~ for $t > T_0 + a$ we can obtain an exponential estimate on the integral. Now for the second integral

$\int_a^t f(x)\, g(t-x)\, dx$ is dominated by

$$M \int_a^t e^{\omega x} |g(t-x)|\, dx$$

$$\leq M e^{\omega t} \int_a^t |g(t-x)|\, dx$$

$$= M e^{\omega t} \int_0^{t-a} |g(u)|\, du$$

$$\leq M e^{\omega t} \int_0^{T_0} |g(u)|\, du + \int_{T_0}^{t-a} |g(u)|\, du$$

The first integral is a finite number and the second integral is dominated by

$$\int_{T_0}^{t-a} \tilde{M} e^{\omega u}\, du \leq C e^{\omega t}$$

Now we show that

$$\mathcal{L}(fH * gH) = (\mathcal{L}f)(\mathcal{L}g)$$

$$\mathcal{L}(fH * gH) = \int_0^\infty e^{-st} (fH * gH)(t)\, dt$$

$$= \int_0^\infty e^{-st}\, dt \int_{-\infty}^{\infty} f(x) H(x)\, g(t-x) H(t-x)\, dx$$

$$= \int_0^\infty e^{-st}\,dt \int_0^\infty f(x)\,g(t-x)\,H(t-x)\,dx$$

$$= \int_0^\infty f(x)\,dx \int_0^\infty e^{-st}\,g(t-x)\,H(t-x)\,dt$$

$$= \int_0^\infty f(x)\,dx \int_x^\infty e^{-st}\,g(t-x)\,dt$$

put $t - x = u$ in the inner integral

$$= \int_0^\infty f(x)\,dx \int_0^\infty e^{-s(x+u)}\,g(u)\,du$$

$$= \int_0^\infty f(x)\,dx \int_0^\infty e^{-sx} \cdot e^{-su} g(u)\,du$$

$$= \int_0^\infty f(x) e^{-sx} \int_0^\infty e^{-su} g(u)\,du$$

$$= (\mathcal{L}f)(\mathcal{L}g)$$ Proof is complete.

<u>Cor:</u> $\Gamma(a)\Gamma(b) = B(a,b)\,\Gamma(a+b)$

<u>pf:</u> We have seen that

$$t^{a-1}H(t) * t^{b-1}H(t) = t^{a+b-1}H(t)B(a,b)$$

$$\mathcal{L}\left(t^{a-1}H(t) * t^{b-1}H(t)\right) = B(a,b)\,\mathcal{L}\left(t^{a+b-1}H(t)\right)$$

Applying the Convolution theorem

$$\frac{\Gamma(a)}{s^a} \cdot \frac{\Gamma(b)}{s^b} = B(a,b)\,\frac{\Gamma(a+b)}{s^{a+b}}$$

$$\therefore \quad \Gamma(a)\Gamma(b) = \Gamma(a+b)\,B(a,b)$$

<u>Exercise:</u> Prove that

$$B(a,a) = 2^{1-2a}\,B\left(a,\tfrac{1}{2}\right)$$

<u>Hint:</u> Make appropriate substitutions in the

beta integral $B(a,a)$

Deduce that

$$\sqrt{\pi}\,\Gamma(a)\,\Gamma(a+\tfrac{1}{2}) = \Gamma(2a)\cdot 2^{1-2a}$$

(This is Legendre's Duplication Formula)

The analogy with the familiar formula

$$2\sin \pi a \sin \pi(a+\tfrac{1}{2}) = \sin 2\pi a$$

is Quite Striking

For the sine function one also has the formula

$$\sin \pi a \sin\left(\pi a + \frac{\pi}{k}\right) \cdots \sin\left(\pi a + \frac{k-1}{k}\pi\right)$$
$$= 2^{1-k} \sin \pi k a$$

Is there an analogue for the Gamma Function?

$$\Gamma(ka) = k^{ka-\frac{1}{2}} (2\pi)^{\frac{1-k}{2}} \Gamma(a)\Gamma\left(a+\frac{1}{k}\right)\cdots\Gamma\left(a+\frac{k-1}{k}\right)$$

This remarkable formula was given by Gauss in 1812 in his great memoir on hypergeometric Series

Disquisitiones generales circa seriem infinitatam $\quad 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \dots$

Solving Integral / Integro differential Eq^{n}s:

Q16 (iii) on p 10

$$\frac{dy}{dt} = 1 - \int_0^t y(t-\tau)\,d\tau \, ; \quad y(0) = 1.$$

Note that the integral is

$$y(t)H(t) * H(t)$$

Laplace transform the equation and we get

$$sY - y(0) = \frac{1}{s} - Y(s)\cdot\frac{1}{s}$$

$$\therefore Y(s)\left(s + \frac{1}{s}\right) = \frac{1}{s} + 1 = \frac{s+1}{s}$$

$Y(s)\,(s^2+1)/s = \dfrac{s+1}{s}$ or

$\therefore \quad Y(s) = \dfrac{s+1}{s^2+1} = \mathcal{L}(\cos t + \sin t)$

$\therefore \quad y(t) = \cos t + \sin t.$

Q16 (ii) $\quad A = \displaystyle\int_0^t \frac{y(x)\,dx}{\sqrt{t-x}}.$

RHS is the Convolution $\quad y(x)H(x) * \dfrac{H(x)}{\sqrt{x}}$

$\therefore \quad \dfrac{A}{s} = Y(s).\sqrt{\dfrac{\pi}{s}}$

$$\therefore \quad Y(s) = \frac{A}{\sqrt{\pi}\cdot\sqrt{s}} = \frac{A}{\pi}\sqrt{\pi/s}$$

or $$y(t) = \frac{A}{\pi\sqrt{t}}.$$

<u>Solve:</u> $$y'' + y = f(t)$$

$$y(0) = c_1, \quad y'(0) = c_2.$$

Laplace transforming

$$s^2 Y - c_2 - sc_1 + Y = F(s)$$

$$(s^2+1)Y - c_2 - sc_1 = F(s)$$

$$\therefore Y = \frac{c_2}{s^2+1} + \frac{s c_1}{s^2+1} + F(s)\,\mathcal{L}\sin t$$

$$\therefore \mathcal{L}y = \mathcal{L}\left(c_1\cos t + c_2\sin t\right) + \mathcal{L}f\,\mathcal{L}\sin t$$

$$= \mathcal{L}(c_1\cos t + c_2\sin t) + \mathcal{L}(fH * (\sin t)H)$$

or $$y(t) = c_1\cos t + c_2\sin t + \int_0^t f(x)\sin(t-x)\,dx$$

Q21 An Integral Computation

$$I = \int_0^\infty \exp\left(-at - \frac{b}{t}\right) \frac{dt}{\sqrt{t}}$$

put $t = \lambda u$

$$I = \int_0^\infty \exp\left(-a\lambda u - \frac{b}{\lambda u}\right) \frac{\lambda\, du}{\sqrt{\lambda}\,\sqrt{u}}$$

$$= \sqrt{\lambda} \int_0^\infty \exp\left(-a\lambda u - \frac{b}{\lambda} u\right) \frac{du}{\sqrt{u}}$$

we choose λ such that

$$a\lambda = \frac{b}{\lambda} = c \text{ say.}$$

This gives $c = \sqrt{ab}$ and

$$\lambda = \sqrt{\frac{b}{a}}$$

$$I = \sqrt{\lambda} \int_0^\infty \exp\left(-cu - \frac{c}{u}\right) \frac{du}{\sqrt{u}}$$

put $u = \frac{1}{v}$

$$I = \sqrt{\lambda} \int_0^\infty \exp\left(-cv - \frac{c}{v}\right) \frac{dv}{v^{3/2}}$$

adding we get

$$2I = \sqrt{\lambda}\int_0^\infty \exp\left(-cu - \frac{c}{u}\right)\left(1+\frac{1}{u}\right)\frac{du}{\sqrt{u}}$$

$$= \sqrt{\lambda}\int_0^\infty \exp -c\left(\left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)^2 \cdot e^{-2c}\right)\left(1+\frac{1}{u}\right)\frac{du}{\sqrt{u}}$$

$$= \sqrt{\lambda}\, e^{-2c}\int_0^\infty \exp -c\left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)^2 \cdot \left(1+\frac{1}{u}\right)\frac{du}{\sqrt{u}}$$

now put $\sqrt{u} - \frac{1}{\sqrt{u}} = w$

$$\left(\frac{1}{2\sqrt{u}} + \frac{1}{2u\sqrt{u}}\right)du = dw$$

$$\because \quad \left(1+\frac{1}{u}\right)\frac{du}{\sqrt{u}} = 2\,dw$$

$$\therefore \quad 2I = 2\sqrt{\lambda}\, e^{-2c} \int_{-\infty}^{\infty} e^{-cw^2}\, dw$$

$$\therefore \quad I = 2\sqrt{\lambda}\, e^{-2c} \int_{0}^{\infty} e^{-cw^2}\, dw$$

$$= \frac{2\sqrt{\lambda}}{\sqrt{c}}\, e^{-2c} \cdot \frac{\sqrt{\pi}}{2}$$

$$I = \sqrt{\frac{\lambda\pi}{c}}\; e^{-2c}$$

Recall $\lambda = \sqrt{\frac{b}{a}}$; $c = \sqrt{ab}$

$$\therefore \quad I = \sqrt{\frac{\pi}{a}}\, e^{-2\sqrt{ab}}$$

Let us now calculate

$$\mathcal{L}\left(\frac{1}{\sqrt{t}}\, e^{-b/t} \right) :$$

$$= \int_0^\infty e^{-st - b/t} \frac{dt}{\sqrt{t}}$$ which as our formula shows,

$$= \sqrt{\frac{\pi}{s}}\, e^{-2\sqrt{sb}}$$

In particular with $b = n^2$; $n \in \mathbb{Z}$

$$\mathcal{L}\left(\frac{1}{\sqrt{t}} e^{-n^2/t}\right) = \sqrt{\frac{\pi}{s}}\, e^{-2|n|\sqrt{s}}$$

Let us sum over $n \in \mathbb{Z}$

$$\mathcal{L}\left(\frac{1}{\sqrt{t}}\left(1 + 2\sum_{n=1}^{\infty} e^{-n^2/t}\right)\right)$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} + \frac{2\sqrt{\pi}}{\sqrt{s}} \sum_{n=1}^{\infty} e^{-2n\sqrt{s}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} + \frac{2\sqrt{\pi}}{\sqrt{s}} \cdot \frac{e^{-2\sqrt{s}}}{1 - e^{-2\sqrt{s}}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} + \frac{\sqrt{\pi}}{\sqrt{s}} \left\{ \frac{1+e^{-2\sqrt{s}}}{1-e^{-2\sqrt{s}}} - \frac{1-e^{-2\sqrt{s}}}{1-e^{-2\sqrt{s}}} \right\}$$

$$= \frac{\sqrt{\pi}\,\coth\sqrt{s}}{\sqrt{s}}$$

Now, in an earlier exercise you were asked to find the Mittag-Leffler expansion for $\cot s$ starting with the Fourier series for $f(x) = x$; $-\pi < x < \pi$

$f(x+2\pi) = f(x)$ for all $x \in \mathbb{R}$.

$$\coth s = \frac{1}{s} + 2\sum_{1}^{\infty} \frac{s}{s^2+n^2\pi^2} \quad \text{(Check)}$$

$$\therefore \frac{\coth\sqrt{s}}{\sqrt{s}} = \frac{1}{s} + 2\sum_{1}^{\infty} \frac{1}{s+n^2\pi^2}$$

$$= \mathcal{L}1 + 2\sum_{1}^{\infty} \mathcal{L}\left(e^{-n^2\pi^2 t}\right)$$

$$= \mathcal{L}\left(1 + 2\sum_{1}^{\infty} e^{-n^2\pi^2 t}\right)$$

Comparing the two results:

$$\sqrt{\pi}\,\mathcal{L}\left(1 + 2\sum_{1}^{\infty} e^{-n^2\pi^2 t}\right) = \mathcal{L}\left(\frac{1}{\sqrt{t}}\left(1 + 2\sum_{n=1}^{\infty} e^{-n^2/t}\right)\right)$$

We have therefore

$$1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t} = \frac{1}{\sqrt{\pi t}}\left(1 + 2\sum_{n=1}^{\infty} e^{-n^2/t}\right)$$

Replacing t by t/π we get the more symmetrical form:

$$1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi t} = \frac{1}{\sqrt{t}}\left(1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi/t}\right)$$

A formula exhibiting Exotic Symmetry!

$$\vartheta(t) = 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi t}$$

is called Jacobi theta function.

We have shown

$$\vartheta(t) = \frac{1}{\sqrt{t}}\,\vartheta\left(\frac{1}{t}\right) \quad \text{(Jacobi 1829)}$$

The most beautiful formula in all of Mathematics – unsurpassed in simplicity and elegance!

Now $$\int_0^\infty e^{-n^2 t}\, t^{s-1}\, dt = \frac{\Gamma(s)}{n^{2s}}$$

Summing over n would give

$$\int_0^\infty t^{s-1}\left(2\sum_{n=1}^{\infty} e^{-n^2 t}\right) dt = \Gamma(s)\,\zeta(2s)$$

$$\zeta(2s) = 1 + \frac{1}{2^{2s}} + \frac{1}{3^{2s}} + \cdots$$

(Riemann Zéta function)

This suggests that the ϑ-function is intimately connected to the ζ-function of Riemann

Indeed it is! $\vartheta(t) = \frac{1}{\sqrt{t}} \vartheta\left(\frac{1}{t}\right)$ is just another "avataar" of the famous Functional Equation of Riemann.

$$\Gamma\left(\frac{s}{2}\right) \zeta(s)\, \pi^{-s/2} = \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)\, \pi^{-(1-s)/2} \quad (s \neq 0, 1)$$

Riemann gave two proofs (1859) one of which starts with the theta function identity of Jacobi.

Oh seeker of beauty,
Come, take a walk in my Garden
But curse me not if you do not find flowers
For beauty sometimes Sleeps upon a blade of grass !

The End! Thank You!!